MATHEMATICS (3), A29
(Preliminary version)
September 28, 2004
Compiled by
Josef Diblík

MATHEMATICS (3), A29

Compiled by

Josef Diblík

MATHEMATICS (3), A29:

13 weeks with 2 hours of lectures & 2 hours of practices

guaranteed by: Prof. RNDr. Josef Diblík, DrSc. *and* Mgr. Veronika Chrastinová, Ph.D.

Programme in 2004/2005:

1. **Fourier Series.**
2. **Multiple Integrals (double, triple).**
3. **Line Integral in Scalar and Vector Fields.**
4. **Surface Integrals.**

Chapter 1

FOURIER SERIES

1.1 Fourier Series and Fourier Transformation

Definition 1. Let $f(x)$ be a function defined in the interval $[-\pi, \pi]$ for which we only suppose that all the integrals below exist. The numbers

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ \ n = 0, 1, 2, \ldots$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx,\ \ n = 1, 2, 3, \ldots$$

are called the **Fourier coefficients** of the function $f(x)$. The series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is called **Fourier series** of the function $f(x)$.

1.2 Dirichlet's Conditions

Definition 2. Function $f(x)$ satisfies on interval $[-\pi, \pi]$ **Dirichlet's conditions** if has only a finite number of maxima and minima in the interval $[-\pi, \pi]$ and is continuous in that interval except a finite number of points of discontinuity of the first kind.

Sometimes are Dirichlet's Conditions formulated as:

Definition 3. Function $f(x)$ satisfies on interval $[-\pi, \pi]$ **Dirichlet's conditions** if this interval can be divided into finite number of subintervals in interior each of them is $f(x)$ monotone and bounded.

Remark 1. By analogy we can formulate Dirichlet's conditions on arbitrary interval $[a, b]$.

1.3 Dirichlet Theorem

Theorem 1. *If the function $f(x)$ satisfies the Dirichlet's conditions then at each point of the interval $[-\pi, \pi]$ the Fourier series converges and*

$$\frac{1}{2}\left[f(x-0)+f(x+0)\right] = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

and at both end points of the interval $[-\pi, \pi]$ the sum of Fourier series equals

$$\frac{1}{2}\left[f(-\pi+0)+f(\pi-0)\right].$$

1.4 Expanding Even and Odd Functions into Fourier Series

1. A) The Case when $f(x)$ is Even

In this situation the functions $f(x)\sin nx$, $n = 1, 2, \ldots$ are odd and, consequently, $b_n = 0$. Therefore the Fourier series has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

since $f(x)\cos nx$ is an even function.

2. B) The Case when $f(x)$ is Odd

In this situation the functions $f(x)\cos nx$, $n = 1, 2, \ldots$ are odd and, consequently, $a_n = 0$. Therefore the Fourier series is written as

$$\sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

since $f(x)\sin nx$ is an even function.

1.5 Expanding Functions with Arbitrary Period

Let function $f(x)$ be defined in an interval $[-l, l]$ where $l > 0$. Then the Fouries series has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,\ \ n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx,\ \ n = 1, 2, 3, \dots .$$

1.6 Expanding Even Function with Arbitrary Period

If function $f(x)$ is even on interval $[-l, l]$ where $l > 0$ then

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,\ \ n = 0, 1, 2, \dots$$

$$b_n = 0,\ \ n = 1, 2, 3, \dots$$

and Fouries series has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\ .$$

1.7 Expanding Odd Function with Arbitrary Period

If function $f(x)$ is odd on interval $[-l, l]$ where $l > 0$ then

$$a_n = 0,\ \ n = 0, 1, 2, \dots$$

$$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx,\ \ n = 1, 2, 3, \dots$$

and Fouries series has the form

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\ .$$

1.8 Expanding Functions into Half-Range Fourier Series

(Methods of Expanding of Functions given on $[0,a]$ into Fourier Series)

Let us consider a function $f(x)$ defined in an interval $[0,a]$, $a > 0$ and suppose that it is required to expand it into a trigonometric series.
The given function $f(x)$ can be extended in an arbitrary fashion to the interval $[-a,0)$ so that new function $F(x)$ defined in the interval $[-a,a]$ and coinciding with $f(x)$ in the interval $[0,a]$ satisfies all requirements of the Dirichlet theorem on $[-a,a]$. On expanding the function $F(x)$ into Fourier's series in the interval $[-a,a]$ we obtain the sought-for trigonometric series representing the original function $f(x)$ in the interval $[0,a]$. Since the extension of $f(x)$ to $F(x)$ can be performed in an arbitrary manner there exist infinitely many such trigonometric series. Some significant cases are considered below.

1. Extension $f(x)$ as an $a-$periodic Function

In this case we put $2l = a \Longrightarrow l = a/2$ and

$$a_n = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\cos\frac{2n\pi x}{a}\,dx,$$
$$b_n = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\sin\frac{2n\pi x}{a}\,dx$$

and Fouries series takes the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{a} + b_n\sin\frac{2n\pi x}{a}\right).$$

2. Extension $f(x)$ as an Even Function

In this case we put $2l = 2a \Longrightarrow l = a$ and

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx,$$
$$b_n = 0$$

and Fouries series takes the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{a}\ .$$

3. Extension $f(x)$ as an Odd Function

Let in this case be put $2l = 2a \Longrightarrow l = a$ and

$$a_n = 0,$$
$$b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

and Fouries series takes the form

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \ .$$

1.9 Example

Let us expand into Fourier's series the function

$$f(x) = \begin{cases} -1, & \text{for} \quad -\pi \le x < 0, \\ \ \ 1, & \text{for} \quad \ \ 0 < x \le \pi. \end{cases}$$

This function is discontinuous at the point $x = 0$ and satisfies the conditions of the Dirichlet theorem. Moreover, since this function is odd, we have $a_n = 0$. Let us determine the coefficients b_n :

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1\cdot\sin nx\,dx = -\frac{2}{n\pi}\cos nx|_0^{\pi} = -\frac{2}{n\pi}(\cos n\pi - 1)\,;$$

$$b_1 = \frac{4}{\pi}\ ,\ b_2 = 0, b_3 = \frac{4}{3\pi}\ ,\ b_4 = 0,\ b_5 = \frac{4}{5\pi}\ ,\ b_6 = 0\,,\ \ldots.$$

Thus

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \ldots\right].$$

1.10 Riemann's Theorem

Theorem 2. *If $f(x)$ satisfies the Dirichlet's condition then:*

$$\lim_{p\to\infty}\int_a^b f(x)\cos px\,dx = 0$$

and

$$\lim_{p\to\infty}\int_a^b f(x)\sin px\,dx = 0.$$

As a consequence of this theorem for coefficients of the Fourier series a_n, b_n it follows:

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{n\to\infty} b_n = 0.$$

Let us show this property on the previous example. It is easy to see that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{-2}{n\pi}(\cos n\pi - 1) = 0$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0.$$

1.11 Fourier Integral

Let us begin with the following theorem concerning integral repsesentation of a given function.

Theorem 3. *If a given function $f(x)$ is defined throughout the $x-$axis and that it is piecewise smooth in any finite interval of the form $[-l, l]$ and, moreover,*

$$\int_{-\infty}^{+\infty} |f(t)|\, dt < \infty \tag{1.1}$$

the the **Fourier's integral formula**

$$f(x) = \frac{1}{\pi}\int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} f(t)\cos\omega(t-x)\, dt \tag{1.2}$$

holds for all the points of continuity of the function $f(x)$. At every point of discontinuity $x = x_0$ Fourier's integral assumes the value

$$\frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)].$$

Remark 2. A function $f(t)$ is said to be **absolutely integrable** or **absolutely summable** if the inequality (1.1) holds.

1.12 Fourier Integral for Even and Odd Functions

Since

$$\cos\omega(t-x) = \cos\omega t\cos\omega x + \sin\omega t\sin\omega x$$

the formula (1.2) can be rewritten in the form

$$f(x) = \frac{1}{\pi}\int_0^{+\infty} \cos\omega x \, d\omega \int_{-\infty}^{+\infty} f(t)\cos\omega t \, dt + \frac{1}{\pi}\int_0^{+\infty} \sin\omega x \, d\omega \int_{-\infty}^{+\infty} f(t)\sin\omega t \, dt.$$

Denoting the inner integrals (together with constant $1/\pi$) as $A(\omega)$ and $B(\omega)$, i.e.

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cos\omega t \, dt$$
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin\omega t \, dt$$

we receive the relation

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x] \, d\omega.$$

Suppose that $f(x)$ is an **even** function. Then the expession

$$f(x)\sin\omega$$

is an odd function and $B(\omega) \equiv 0$. Besides, in this case

$$A(\omega) = \frac{2}{\pi}\int_0^{+\infty} f(t)\cos\omega t \, dt$$

and, consequently,

$$f(x) = \frac{2}{\pi}\int_0^{+\infty} \cos\omega x \, d\omega \int_0^{\infty} f(t)\cos\omega t \, dt.$$

Similarly, if $f(x)$ is an **odd** function, we obtain $A(\omega) \equiv 0$ and

$$f(x) = \frac{2}{\pi}\int_0^{+\infty} \sin\omega x \, d\omega \int_0^{\infty} f(t)\sin\omega t \, dt.$$

1.13 Fourier's Cosine and Sine Transform

Let the function $f(x)$ be **even.** Then so called **Fourier's cosine transform** is defined by formula

$$F_C(\omega) = 2\int_0^{+\infty} f(t)\cos\omega t \, dt$$

and so called **inverse Fourier's cosine transform** is defined by formula

$$f(x) = \frac{1}{\pi} \int_0^\infty F_C(\omega) \cos \omega x \, d\omega.$$

Let the function $f(x)$ be **odd.** Then so called **Fourier's sine transform** is defined by formula

$$F_S(\omega) = 2 \int_0^{+\infty} f(t) \sin \omega t \, dt$$

and so called **inverse Fourier's sine transform** is defined by formula

$$f(x) = \frac{1}{\pi} \int_0^\infty F_S(\omega) \sin \omega x \, d\omega.$$

Example 1. Let us represent as Fourier's integral the function

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 \le x < 1, \\ 0 & \text{for} \quad x > 1 \end{cases}$$

by extending it as an even function and then as an odd function.

Solution. In the former case we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \, d\omega \int_0^1 \cos \omega t \, dt = \frac{2}{\pi} \int_0^\infty \frac{\cos \omega x \sin \omega}{\omega} \, d\omega.$$

By the way, this result shows that

$$\frac{2}{\pi} \int_0^\infty \frac{\cos \omega x \sin \omega x}{\omega} \, d\omega = \begin{cases} 1 & \text{for} \quad -1 < x < 1, \\ \frac{1}{2} & \text{for} \quad x = \pm 1 \\ 0 & \text{for} \quad x < -1 \wedge x > 1. \end{cases}$$

In the latter case (the odd extension) we obtain

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \omega x \, d\omega \int_0^1 \sin \omega t \, dt = \frac{2}{\pi} \int_0^\infty \frac{\sin \omega x \cdot (1 - \cos \omega)}{\omega} \, d\omega.$$

Example 2. Let us construct Fourier's integrals for the function $f(x) = e^{-\beta x}$, $\beta > 0, x > 0$ extended as in the previous example, in both ways, to the negative half-axis Ox.

Solution. For the even extension we receive

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \, d\omega \int_0^\infty e^{-\beta t} \sin \omega t \, dt = \dots = \frac{2}{\pi} \int_0^\infty \frac{\omega \cdot \cos \omega x}{\omega^2 + \beta^2} \, d\omega$$

and for the odd extension

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \omega x \, d\omega \int_0^\infty e^{-\beta t} \sin \omega t \, dt = \dots = \frac{2}{\pi} \int_0^\infty \frac{\omega \cdot \sin \omega x}{\omega^2 + \beta^2} \, d\omega.$$

1.14 Fourier Integral in Complex Form

Replacing $\cos\omega x, \sin\omega x$ in the Fourier integral by their expressions given by Euler's formulas we derive

$$f(x) = \int_0^\infty \left[\frac{1}{2}A(\omega)\left(e^{i\omega x} + e^{-i\omega x}\right) + \frac{1}{2i}B(\omega)\left(e^{i\omega x} - e^{-i\omega x}\right)\right] d\omega =$$
$$\frac{1}{2}\int_0^\infty \left\{[A(\omega) - iB(\omega)]e^{i\omega x} + [A(\omega) + iB(\omega)]e^{-i\omega x}\right\} d\omega.$$

Let us put

$$A(\omega) - iB(\omega) = \frac{1}{\pi}F(\omega).$$

Then

$$F(\omega) = \int_{-\infty}^{+\infty} [\cos\omega t - i\sin\omega t]\, dt = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\, dt.$$

Furthermore, we have

$$A(\omega) + iB(\omega) = \frac{1}{\pi}\overline{F(\omega)} = \frac{1}{\pi}F(-\omega)$$

and

$$f(x) = \frac{1}{2\pi}\int_0^\infty \left[F(\omega)e^{i\omega x} + F(-\omega)e^{-i\omega x}\right] d\omega =$$
$$\frac{1}{2\pi}\int_0^\infty F(\omega)e^{i\omega x}\, d\omega - \frac{1}{2\pi}\int_0^{-\infty} F(\omega)e^{i\omega x}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)e^{i\omega x}\, d\omega.$$

Therefore

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)e^{i\omega x}\, d\omega.$$

1.15 Fourier Transform

The function

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^\infty f(t)e^{-i\omega t}\, dt$$

is called the **Fourier transform** of the function $f(t)$; the **original function**

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty F(\omega)e^{i\omega t}\, d\omega$$

is called **Fourier's inverse transform** of the function $F(\omega)$. Fourier transform $F(\omega)$ is also called the **spectral function** of $f(t)$. The function

$|F(\omega)|$ is termed the **amplitude spectrum** of $f(t)$ and the expression $\varphi(\omega) = -\arg F(\omega)$ the **phase spectrum.**

Example 3. Let us find the Fourier transform and the amplitude and phase spectra for the function

$$f(t) = \begin{cases} e^{-at}, a > 0 & \text{for} \quad t > 0, \\ \quad 0 & \text{for} \quad t < 0. \end{cases}$$

Solution. We have

$$F(\omega) = \int_0^\infty e^{-at} \cdot e^{-i\omega t}\, dt = -\left. \frac{e^{-t(a+i\omega)}}{a + i\omega} \right|_0^\infty = \frac{1}{a + i\omega}\ .$$

Consequently,

$$|F(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

and

$$\varphi(\omega) = -\arg F(\omega) = \arctan \frac{\omega}{a}\ .$$

Chapter 2

MULTIPLE INTEGRALS

2.1 The Volume of a Curvilinear Cylinder

Let us denote the volume of a spatial body by V. Now we break up the base D of the cylindroid into subdomains by means of two systems of coordinate lines $x = \text{const}$ and $y = \text{const}$.

Area of each of subdomain D_{ij} (suppose $D_{ij} \subset D$) is equal to $\Delta x_i \cdot \Delta y_j$ where $\Delta x_i = x_{i+1} - x_i$, $\Delta y_j = y_{j+1} - y_j$. The volume is approximatively equal to the **integral sum,** i.e

$$V \approx \sum_{i=0,j=0}^{n-1,m-1} f(\xi_i, \xi_j) \Delta x_i \Delta y_j.$$

Let us denote $\Delta = \max_{i,j}\{\Delta x_i, \Delta y_j\}$. Then for the volume V we have exact expression

$$V = \lim_{n,m\to\infty,\Delta\to 0} \sum_{i=0,j=0}^{n-1,m-1} f(\xi_i, \xi_j) \Delta x_i \Delta y_j$$

provided that the last limit exists.

2.2 Definition of the Double Integral

Definition 4. The limit of integral sum (provided it exists)

$$\lim_{n,m\to\infty,\Delta\to 0} \sum_{i=0,j=0}^{n-1,m-1} f(\xi_i, \xi_j) \Delta x_i \Delta y_j$$

is called the double integral of the function $f(x,y)$ over the domain D. This is written as

$$\iint_D f(x,y)\,dxdy = \lim_{n,m\to\infty,\Delta\to 0} \sum_{i=0,j=0}^{n-1,m-1} f(\xi_i,\xi_j)\Delta x_i \Delta y_j.$$

The geometrical meaning of double integral was given in the previous section.

2.3 Properties of Double Integrals

1. The double integral of a sum (or difference) of two functions is equal to the sum (or difference) of the double integrals of the summands:

$$\iint_D [f(x,y) \pm \varphi(x,y)]\,dxdy = \iint_D f(x,y)\,dxdy \pm \iint_D \varphi(x,y)\,dxdy.$$

2. A constant factor in the integrand can be taken outside the symbol of the double integral:

$$\iint_D Cf(x,y)\,dxdy = C\iint_D f(x,y)\,dxdy.$$

3. If the domain of integration D is split into two domains D_1 and D_2 having no **interior points** in common then

$$\iint_D f(x,y)\,dxdy = \iint_{D_1} f(x,y)\,dxdy + \iint_{D_2} f(x,y)\,dxdy.$$

4. If two functions $f(x,y)$ and $\varphi(x,y)$ satisfy the condition

$$f(x,y) \geq \varphi(x,y)$$

at all the points of the domain of integration D then

$$\iint_D f(x,y)\,dxdy \geq \iint_D \varphi(x,y)\,dxdy.$$

5.

$$\iint_D dxdy = S$$

where S is the area of domain D.

6. If

$$m \leq f(x,y) \leq M$$

for $(x,y) \in D$ then

$$mS \leq \iint_D f(x,y)\,dxdy \leq MS.$$

7. If $f(x,y) \in C(D)$ then there is a point $(\xi,\nu) \in D$ such that

$$\iint_D f(x,y)\,dxdy = f(\xi,\nu) \cdot S$$

(this formula is called **the mean value theorem** for the double integral) or

$$f(\xi,\nu) = \frac{1}{S}\iint_D f(x,y)\,dxdy.$$

8.

$$\left|\iint_D f(x,y)\,dxdy\right| \leq \iint_D |f(x,y)|\,dxdy.$$

2.4 Evaluating Double Integrals

Definition 5. An elementary domain of the first type D_1 is defined as

$$D_1 = \{(x,y) \in \mathbb{R}^2, a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}.$$

An elementary domain of the second type D_2 is defined as

$$D_2 = \{(x,y) \in \mathbb{R}^2, a \leq y \leq b, \varphi_1(y) \leq x \leq \varphi_2(y)\}.$$

We remind that the problem of computing the volume of a solid was already considered in connection with the applications of the definite integral to geometrical problems. We derived the formula

$$V = \int_a^b S(x)\,dx$$

for the volume of a solid where $S(x)$ is the cross-section area in the plane perpendicular to the axis abscissas cutting that axis at the point x while

$x = a$ and $x = b$ are the equations of the planes bounding the given solid. Since

$$S(x^*) = \int\limits_{f_1(x^*)}^{f_2(x^*)} f(x^*, y)\, dy$$

we get

$$V = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x, y)\, dy \right] dx.$$

We have so called **Fubini** theorem (for double integral):

$$\iint\limits_{D_1} f(x, y)\, dxdy = \int_a^b dx \int_{f_1(x)}^{f_2(x)} f(x, y)\, dy.$$

By analogy we can obtained

$$\iint\limits_{D_2} f(x, y)\, dxdy = \int_a^b dy \int_{\varphi_1(y)}^{\varphi_2(y)} f(x, y)\, dx.$$

These formulas show that the computation of the double integral reduces to two consecutive ordinary definite integrations; one should memorize that in the inner integral one of the variables is considered constant in the integration process. The expansions on the right-hand sides of these formulas are called (twofold) iterated or repeated integrals, tho whole computation process being referred to as the reduction of the double integral to an iterated integral.

The reduction of a double integral to an iterated integral becomes particularly simple when the domain of integration D is a rectangle with sides parallel to the coordinate axes. In this case not only the limits of integration in the outer integral are constant but those in the inner integral as well:

$$\iint\limits_{D} f(x, y)\, dxdy = \int_a^b dx \int_c^d f(x, y)\, dy = \int_c^d dy \int_a^b f(x, y)\, dx.$$

Example 4. Let us find the volume V of the solid bounded by the surface $z = 1 - 4x^2 - y^2$ and by the plane Oxy.
Solution. The solid is a segment of the elliptical paraboloid lying above the plane Oxy. The paraboloid cuts the xy-plane along the ellipse $4x^2 + y^2 = 1$. The problem thus reduces to computing the volume of the **cylindroid**

without lateral cylindrical surface bounded above by the paraboloid $z = 1 - 4x^2 - y^2$ and having the interior of the ellipse as base. The solid under consideration being symmetric with respect to the planes Oxz and Oyz, it is sufficient to determine the quarter volume lying in the first coordinate trihedral. The latter is equal to the double integral over the domain specified by the conditions

$$4x^2 + y^2 \le 1,\ x \ge 0,\ y \ge 0,$$

i.e. over the quarter ellipse. Integrating with respect to y and then with respect to x we receive

$$\frac{1}{4}V = \int_0^{1/2} dx \int_0^{\sqrt{1-4x^4}} (1 - 4x^2 - y^2)\, dy =$$

$$= \int_0^{1/2} dx \left[y - 4x^2 y - \frac{1}{3}y^3 \right] \Bigg|_0^{\sqrt{1-4x^4}} =$$

$$= \int_0^{1/2} \left[(1-4x^2)^{\frac{3}{2}} - \frac{1}{3}(1-4x^2)^{\frac{3}{2}} \right] dx = \frac{2}{3}\int_0^{1/2} \left[1 - 4x^2 \right]^{\frac{3}{2}} dx =$$

$$= \left\{ x = \frac{1}{2}\sin t \right\} = \frac{2}{3}\cdot\frac{1}{2}\int_0^{\pi/2} [1 - \sin^2 t]^{\frac{3}{2}} \cos t\, dt = \frac{1}{3}\int_0^{\pi/2} \cos^4 t\, dt =$$

$$= \left\{ \cos^2 t = \frac{1}{2}(1 + \cos 2t),\ \cos^4 t = \frac{1}{4}\left[1 + 2\cos 2t + \frac{1}{2}(1 + \cos 4t) \right] \right\} =$$

$$= \frac{1}{12}\int_0^{\pi/2} \left[\frac{3}{2} + 2\cos 2t + \frac{1}{2}\cos^4 t \right] dt = \frac{1}{12}\left[\frac{3}{2} +\ \sin 2t \Big|_0^{\pi/2} + \frac{1}{8}\sin 4t \Big|_0^{\pi/2} \right] =$$

$$= \frac{3\pi}{48} = \frac{\pi}{16}\ .$$

Whence $V = \pi/4$.

2.5 Change of Variables in the Double Integrals

Theorem 4. *If the variables x and y in the double integral are replaced by some new variables u and v given by formulas*

$$\begin{aligned} x &= x(u, v) \\ y &= y(u, v) \end{aligned} \tag{2.1}$$

the formula for change of variables has the form

$$\iint_D f(x, y)\, dxdy = \iint_G f[x(u, v), y(u, v)] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv \tag{2.2}$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

and G is transformation of domain D by means of formulas (2.1).

Remark 3. The expression J is a (Jacobi's) functional determinant (a Jacobian).

2.6 The Double Integral in Polar Coordinates

Let us apply the general formula to the transformation from Cartesian coordinates (x and y) to **polar coordinates** (which we denote r and φ instead of u and v):

$$x = r\cos\varphi, \quad y = r\sin\varphi.$$

It is assumed that $r \geq 0$ and that the angle φ ranges from 0 to $2\pi(\varphi \in [0, 2\pi))$.

The Jacobian of this mapping is

$$J = \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \varphi} - \frac{\partial x}{\partial \varphi} \cdot \frac{\partial y}{\partial r} = \cos\varphi \cdot r \cdot \cos\varphi - (-r\sin\varphi) \cdot \sin\varphi = r.$$

Therefore

$$\iint_D f(x,y)\,dxdy = \iint_G f(r\cos\varphi, r\sin\varphi) \cdot r\,drd\varphi.$$

Example 5. Find

$$I = \iint_D \sqrt{x^2+y^2}\,dxdy$$

where

$$D = \{(x,y) \in \mathbb{R}^2, 0 \leq x,\ 0 \leq y,\ x^2+y^2 \leq a^2\}$$

and a is a positive constant. For polar coordinates we have

$$\left.\begin{array}{l} 0 \leq x = r\cos\varphi \\ 0 \leq y = r\sin\varphi \end{array}\right\} \Longrightarrow \left.\begin{array}{l} \cos\varphi \geq 0 \\ \sin\varphi \geq 0 \end{array}\right\} \Longrightarrow 0 \leq \varphi \leq \frac{\pi}{2}$$

and, moreover, $x^2+y^2 = r^2 \leq a^2 \Longrightarrow r \leq a$. Therefore for G we have the definition:

$$G = \left\{(r,\varphi) : 0 \leq r \leq a,\ 0 \leq \varphi \leq \frac{\pi}{2}\right\}.$$

At the end

$$I = \iint_G \sqrt{r^2} \cdot r\, dr d\varphi = \iint_G r^2\, dr d\varphi = \int_0^a r^2\, dr \int_0^{\pi/2} d\varphi = \frac{\pi}{2} \cdot \frac{a^3}{3} = \frac{\pi a^3}{6} \ .$$

Remark 4. In some cases it is useful to use so called generalized polar coordinates:

$$x = ar \cos k\varphi,$$
$$x = br \sin k\varphi, \quad k, a, b \in \mathbb{R}.$$

2.7 Triple Integrals

Let a function $u = f(, x, y, z)$ be defined on a set

$$D \subset \{(x, y, z) \in \mathbb{R}^3, a \le x \le b, c \le y \le d, e \le z \ \le f\}.$$

Let us break up intervals $[a, b], [c, d], [e, f]$ into subintervals:

$$[a, b] = \cup_{i=1}^{n-1} [x_i, x_{i+1}]; \text{ where } a = x_1 < x_2 < \cdots < x_n = b,$$
$$[c, d] = \cup_{j=1}^{m-1} [y_j, y_{j+1}]; \text{ where } c = y_1 < y_2 < \cdots < y_n = d,$$
$$[e, f] = \cup_{k=1}^{o-1} [z_k, z_{k+1}]; \text{ where } e = z_1 < z_2 < \cdots < z_n = f.$$

Let us define subdomains

$$D_{ijk} = \{(x, y, z) : x_i \le x \le x_{i+1}, y_j \le y \le y_{j+1}, z_k \le z \ \le z_{k+1}\}$$

where $i \in \{1, 2, \ldots, n-1\}, j \in \{1, 2, \ldots, m-1\}, k \in \{1, 2, \ldots, o-1\}$ and we will consider only such subdomains D_{ijk} which are a subset of D : $D_{ijk} \subset D$. Chose in each subdomain D_{ijk} a point

$$(\xi_i, \xi_j, \xi_k) \in D_{ijk}$$

and let us define the number

$$\Delta = \max_{i,j,k} (\Delta x_i, \Delta y_j, \Delta z_k).$$

Definition 6. The triple integral of the function $f(x, y, z)$ over the domain D is defined as limit of integral sum (provided it exists), i.e.

$$\iiint_D f(x, y, z)\, dx dy dz = \lim_{\substack{n, m, o \to \infty \\ \Delta \to 0}} \sum_{i,k,j=0}^{n-1, m-1, o-1} f(\xi_i, \xi_k, \xi_k) \Delta x_i \Delta y_j \Delta z_k.$$

Remark 5. The properties of the double integral are extended without any essential changes to the triple integrals.

2.8 Geometrical and Physical Meaning of the Triple Integral

a) Geometrical meaning. If the integrand $f(x, y, z)$ is identically equal to unity, the triple integral expresses the volume V of the domain D :

$$V = \iiint_D dxdydz.$$

a) Physical meaning. Let us consider a body occupying a spatial domain D. We shall suppose that the density of mass distribution within the body is a known function continuous throughout $D : \delta = \delta(x, y, z)$ (kg/m^3). The total mass M of the nonhomogeneous body D is equal to

$$M = \iiint_D \delta(x, y, z)\, dxdydz.$$

2.9 Evaluating Triple Integral

Let us introduce **elementary domains** for evaluating

$$M_1\{(x, y, z) : \ a \le x \le b, f_1(x) \le y \le f_2(x),\ F_1(x, y) \le z \le F_2(x, y)\},$$
$$M_2\{(x, y, z) : \ a \le x \le b, f_1^0(x) \le z \le f_2^0(x),\ F_1^0(x, z) \le y \le F_2^0(x, z)\},$$
$$M_3\{(x, y, z) : \ a \le y \le b, \varphi_1(y) \le x \le \varphi_2(y),\ \Phi_1(x, y) \le z \le \Phi_2(x, y)\},$$
$$M_4\{(x, y, z) : \ a \le y \le b, \varphi_1^0(y) \le z \le \varphi_2^0(y),\ \Phi_1^0(y, z) \le x \le \Phi_2^0(y, z)\},$$
$$M_5\{(x, y, z) : \ a \le z \le b, \omega_1(z) \le x \le \omega_2(z),\ \Omega_1(x, z) \le y \le \Omega_2(x, z)\},$$
$$M_6\{(x, y, z) : \ a \le z \le b, \omega_1^0(z) \le y \le \omega_2^0(z),\ \Omega_1^0(y, z) \le x \le \Omega_2^0(y, z)\}.$$

Theorem 5. Fubini theorems *(for triple integral):*

$$\iiint_{M_1} f(x, y, z)\, dxdydz = \int_a^b dx \int_{f_1(x)}^{f_2(x)} dy \int_{F_1(x,y)}^{F_2(x,y)} f(x, y, z)\, dz,$$
$$\iiint_{M_2} f(x, y, z)\, dxdydz = \int_a^b dx \int_{f_1^0(x)}^{f_2^0(x)} dz \int_{F_1^0(x,z)}^{F_2^0(x,z)} f(x, y, z)\, dy$$

etc.

Example 6. Let us evaluate the triple integral

$$I = \iiint_D (x + y + z)\, dxdydz$$

over the domain D bounded by the coordinate planes $x = 0$, $y = 0$, $z = 0$ and by the plane $x + y + z = 1$.

Solution. The domain D can be written in the form

$$D = \{(x, y, z) : 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Therefore

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (x + y + z)\, dz =$$

$$= \int_0^1 dx \int_0^{1-x} dy \left[(x + y)z + \frac{1}{2} z^2 \right] \Big|_0^{1-x-y} =$$

$$= \int_0^1 dx \int_0^{1-x} \left[(x + y) - (x + y)^2 + \frac{1}{2}(1 - x - y)^2 \right] dy =$$

$$= \int_0^1 dx \left[\frac{1}{2}(x + y)^2 - \frac{1}{3}(x + y)^3 - \frac{1}{6}(1 - x - y)^3 \right] \Big|_0^{1-x} =$$

$$= \int_0^1 \left[\frac{1}{2} - \frac{1}{2}x^2 - \frac{1}{3} + \frac{1}{3}x^3 + \frac{1}{6}(1 - x)^3 \right] dx =$$

$$= \frac{1}{2} - \frac{1}{6} - \frac{1}{3} + \frac{1}{12} - \frac{1}{24}(1 - x)^4 \Big|_0^1 =$$

$$= \frac{1}{2} - \frac{1}{6} - \frac{1}{3} + \frac{1}{12} + \frac{1}{24} = \frac{1}{24}[12 - 4 - 8 + 2 + 1] = \frac{1}{8}\ .$$

2.10 Change of Variables in the Triple Integral

Theorem 6. *If $x = x(u, v, w)$, , $y = y(u, v, w)$, $z = z(u, v, w)$ and the domain D is transformed into G then*

$$\iiint_D f(x, y, z)\, dxdydz = \iiint_G f[x(u, v, w), y(u, v, w), z(u, v, w)] |J|\, dudvdw$$

where

$$J = \begin{vmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{vmatrix}\ .$$

2.11 Cylindrical Coordinates

Cylindrical coordinates are defined by means of formulas

$$x = r\cos\varphi,$$

$$y = r\sin\varphi,$$

$$z = z,$$

where

$$0 \le r, 0 \le \varphi < 2\pi, z \in \mathbb{R}.$$

Then $M(x, y, z) = M(r, \varphi, z)$ and Jacobian

$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

In this case

$$\iiint_D f(x, y, z)\, dxdydz = \iiint_G f[r\cos\varphi, r\sin\varphi, z] r\, drd\varphi dz.$$

Example 7. Let us evaluate

$$I = \iiint_D z\, dxdydz,$$

where

$$D = \{(x, y, z) : x^2 + y^2 \le z, x^2 + y^2 + z^2 \le 6\}.$$

This domain is the space between paraboloid and sphere. Since $r^2 \le z, r^2 + z^2 \le 6$ then an equation of the intersection curve of above surfaces is given by relation

$$r^4 = 6 - r^2 \Longrightarrow (r^2 - 2)(r^2 + 3) = 0 \Longrightarrow r = \sqrt{2},\ z = 2$$

and

$$G = \{(r, \varphi, z) : 0 \le \varphi < 2\pi, 0 \le r \le \sqrt{2}, r^2 \le z \ \le \sqrt{6 - r^2}\}.$$

Therefore

$$I = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} r\, dr \int_{r^2}^{\sqrt{6-r^2}} z\, dz =$$

$$= 2\pi \int_0^{\sqrt{2}} \frac{r}{2}\ z^2\Big|_{r^2}^{\sqrt{6-r^2}}\ dr = \pi \int_0^{\sqrt{2}} r\left[6 - r^2 - r^4\right]\, dr =$$

$$= \pi\left[\frac{6r^2}{2} - \frac{r^4}{4} - \frac{r^5}{5}\right]\Bigg|_0^{\sqrt{2}} = \pi\left[3\cdot 2 - 1 - \frac{8}{6}\right] = \pi\left[6 - 1 - \frac{4}{3}\right] = \frac{11\pi}{3}\ .$$

2.12 Spherical Coordinates

Spherical coordinates are defined by means of formulas

$$x = r\sin\psi\cos\varphi,$$
$$y = r\sin\psi\sin\varphi,$$
$$z = r\cos\psi,$$

where

$$0 \leq r, 0 \leq \psi < \pi, 0 \leq \varphi < 2\pi.$$

Then $M(x, y, z) = M(r, \varphi, \psi)$ and Jacobian

$$J = \begin{vmatrix} \sin\psi\cos\varphi & -r\sin\psi\sin\varphi & r\cos\psi\cos\varphi \\ \sin\psi\sin\varphi & r\sin\psi\cos\varphi & r\cos\psi\sin\varphi \\ \cos\psi & 0 & -r\sin\varphi \end{vmatrix} =$$

$$= r^2(-\sin^3\psi\cos^2\varphi - \sin\psi\cos^2\psi\sin^2\varphi - \cos^2\psi\sin\psi\cos^2\varphi - \sin^3\psi\sin^2\varphi) =$$

$$= r^2(-\sin\psi\sin^2\psi - \sin\psi\cos^2\psi) = -r^2\sin\psi.$$

In this case

$$\iiint_D f(x, y, z)\, dxdydz =$$

$$= \iiint_G f[r\sin\psi\cos\varphi, r\sin\psi\sin\varphi, r\cos\psi]r^2\sin\psi\, drd\varphi d\psi.$$

Example 8. Let us evaluate

$$I = \iiint_D x^2\, dxdydz,$$

where

$$D = \{(x, y, z) : x^2 + y^2 + z^2 \leq R^2\}.$$

This domain can be written in spherical coordinates as $r^2 \leq R^2$. Therefore

$$I = \int_0^R dr \int_0^{2\pi} d\varphi \int_0^{\pi} r^2 \sin^2\psi \cos^2\varphi \cdot r^2 \sin\psi \, d\psi =$$

$$= \left. \frac{r^5}{5} \right|_0^R \int_0^{2\pi} \cos^2\varphi \, d\varphi \int_0^{\pi} \sin^3\psi \, d\psi =$$

$$= \frac{R^5}{5} \int_0^{2\pi} \frac{1+\cos 2\varphi}{2} \, d\varphi \int_0^{\pi} (1-\cos^2\psi) \sin\psi \, d\psi = [t = \cos\psi] =$$

$$= -\frac{R^5}{10} \left. \left(\varphi - \frac{1}{2}\sin 2\varphi\right) \right|_0^{2\pi} \int_1^{-1} (1-t^2) \, dt = -\frac{2\pi R^5}{10} \left. \left(t - \frac{t^3}{3}\right) \right|_1^{-1} =$$

$$= -\frac{2\pi R^5}{10} \left(-1 + \frac{1}{3} - 1 + \frac{1}{3}\right) = -\frac{2\pi R^5}{10} \left(-2 + \frac{2}{3}\right) = \frac{4\pi R^5}{15} \; .$$

Chapter 3

LINE INTEGRAL IN SCALAR AND VECTOR FIELDS

3.1 Line Integrals - Motivation. Mass of the Wire

Let us imagine a thin wire shaped like the smooth curve C with endpoints A and B. Suppose that the wire has variable density given at the point (x, y, z) by the known continuous function $f(x, y, z)$, in units such as grams per (linear) centimeter. Let

$$x = x(t), y = y(t), z = z(t)$$

be a smooth parameterization of the curve C, with $t = a$ corresponding to the initial point A of the curve and $t = b$ to its terminal point B. To approximate the total mass m of the curved wire, we begin with a partition

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

of $[a, b]$ into n subintervals. These subdivision points of $[a, b]$ produce, via our parameterization, a physical division of the wire into short curve segments. We denote as P_i the point

$$(x(t_i), y(t_i), z(t_i)),\ i = 0, 1, \ldots, n.$$

Then we can approximate the mass of the wire:

$$m \approx \sum_{i=1}^{n} f(\xi_i)\Delta s_{i-1}$$

where Δs_i is the length (always **positive**) of the segment of the curve C between the points P_{i-1}, P_i. The limit of this sum as $\Delta t \to 0$ (or as

$\Delta s \to 0$) should be the actual mass m. This is our motivation for the definition of the line integral of the function f along the curve C denoted by

$$m = \int_C f(x, y, z)\, ds = \lim_{n\to\infty, \Delta\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta s_{i-1}.$$

3.2 The Line Integral (of the First Type)

Definition 7. (Line integral with respect to the arc length; line integral of the first type.)
Suppose that $f(x, y, z)$ is continuous at each point of the smooth parametric curve C from A to B. Then the line integral of f along C from A to B with respect to arc length is defined to be

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty, \Delta\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta s_{i-1}.$$

3.3 Evaluation of the Line Integral of the First Type

If

$$x = x(t), y = y(t), z = z(t)$$

be a smooth parameterization of the curve C, with $t = a$ corresponding to the initial point A of the curve and $t = b$ to its terminal point B then

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

This is an ordinary integral with respect to the single real variable t. In the case when $z \equiv 0$ (curve C lies in the $xy-$plane) we have

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Example 9. Evaluate the line integral

$$I = \int_C xy\, ds$$

where C is the first quadrant quarter-circle parameterized by

$$x = \cos t, y = \sin t,\ 0 \le t \le \frac{\pi}{2}\,.$$

Solution. By previous formula we can see that

$$I = \int_C xy\,ds = \int_0^{\pi/2} \cos t \sin t \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt =$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin 2t\,dt = \frac{1}{2}\left.\frac{-\cos 2t}{2}\right|_0^{\pi/2} = \frac{1}{4}(1+1) = \frac{1}{2}.$$

3.4 Line Integrals with Respect to Coordinate Variables (Line Integral of the Second Type)

The line integral of f along C with respect to x is defined to be

$$\int_C f(x,y,z)dx = \lim_{n\to\infty,\Delta\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_{i-1}$$

(Δx_i **can not preserves the sign**). Thus

$$\int_C f(x,y,z)dx = \int_a^b f(x(t),y(t),z(t))x'(t)dt.$$

Similarly, the line integrals of f along C with respect to y and with respect to z are given by

$$\int_C f(x,y,z)dy = \int_a^b f(x(t),y(t),z(t))y'(t)dt$$

and

$$\int_C f(x,y,z)dz = \int_a^b f(x(t),y(t),z(t))z'(t)dt.$$

The last three integrals typically occur together. If P, Q and R are continuous functions of the variables x, y and z then we define **the line integral of the second type** as

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz.$$

Example 10. Evaluate the integral

$$I = \int_C ydx + zdy + xdz$$

where C is the parametric curve $x = t, y = t^2, z = t^3, 0 \le t \le 1$.

Solution. We have

$$I = \int_0^1 (t^2 + t^3\cdot 2t + t\cdot 3t^2)\,dt = \int_0^1 (t^2 + 3t^3 + 2t^4)\,dt =$$

$$= \frac{1}{3} + \frac{3}{4} + \frac{2}{5} = \frac{1}{60}(20+45+24) = \frac{89}{60}\,.$$

3.5 Difference Between the Line Integral of the First and Second Type

Suppose that the orientation of the curve C (the direction in which it is traced as t increases) is reversed. Then, because of the terms $x'(t)$, $y'(t)$, $z'(t)$ the sign of the line integral of the second type is changed. But this reversal of orientation does not change the value of the line integral of the first type. We may express this by writing

$$\int_{C^-} f\, ds = \int_C f\, ds$$

in contrast with the formula

$$\int_{C^-} Pdx + Qdy + Rdz = -\int_C Pdx + Qdy + Rdz$$

where the symbol C^- denotes the curve C with its orientation reserved (from B to A rather than from A to B).

3.6 Line Integrals and Work

Let us suppose that

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

is a force field defined on a region that contains the curve C. Suppose that C has a parameterization

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \quad t \in [a, b]$$

with a nonzero velocity vector

$$\vec{v}(t) = \frac{dx(t)}{dt}\vec{i} + \frac{dy(t)}{dt}\vec{j} + \frac{dz(t)}{dt}\vec{k}.$$

The speed associated with this velocity vector is

$$v(t) = |\vec{v}(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,.$$

Unit tangent vector to the curve C is equal to

$$\vec{T(t)} = \frac{\vec{v}(t)}{v(t)} = \frac{1}{v(t)}(x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}).$$

We want to approximate the work W done by the force $\vec{F}$ in moving a particle along the curve C from A to B. Let us subdivide C as indicated. Think of $\vec{F}$ moving the particle from P_{i-1} to P_i. The work ΔW_i done is approximately the product of distance Δs_i from P_{i-1} to P_i (measured along C) and the tangential component $\vec{F} \cdot \vec{T}$ of the force $\vec{F}$ at a typical point

$$(x(t_i^*), y(t_i^*), z(t_i^*))$$

between P_{i-1} and P_i. Thus

$$\Delta W_i \approx \vec{F}(x(t_i^*), y(t_i^*), z(t_i^*)) \cdot \vec{T}(t_i^*) \Delta s_i$$

so that total work W is given approximately by

$$W \approx \sum_{i=1}^{n} \Delta W_i = \sum_{i=1}^{n} \vec{F}(x(t_i^*), y(t_i^*), z(t_i^*)) \cdot \vec{T}(t_i^*) \Delta s_i.$$

This approximation suggest that we define the work W as

$$W = \int_C \vec{F} \cdot \vec{T}\, ds.$$

It is customary to write formally

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k},\ d\vec{r} = \vec{i}dx + \vec{j}dy + \vec{k}dz$$

and

$$\vec{T}\, ds = d\vec{r}.$$

Then

$$W = \int_C \vec{F} \cdot d\vec{r}.$$

To evaluate W we express its integrand in terms of the parameter t, as usual.

$$W = \int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b (P\vec{i} + Q\vec{j} + R\vec{k})(x'\vec{i} + y'\vec{j} + z'\vec{k})\, dt =$$

$$= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt = \int_a^b (Pdx + Qdy + Rdz)\,.$$

Therefore

$$W = \int_a^b Pdx + Qdy + Rdz = \int_C Pdx + Qdy + Rdz.$$

3.7 Independence of Line Integrals of Path

Theorem 7. *The line integral*

$$W = \int_C \vec{F} \cdot \vec{T} \, ds$$

is independent of path if and only if

$$\vec{F} = \nabla f$$

for some function f.

Let us proof the one part of this theorem only. Suppose

$$\vec{F} = \nabla f = (f'_x, f'_y, f'_z)$$

and C is a path from A to B parameterized as usual with parameter t in $[a, b]$. Then

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_a^b \left(f'_x dx + f'_y dy + f'_z dz \right) =$$

$$= \int_a^b \left(f'_x \frac{dx}{dt} + f'_y \frac{dy}{dt} + f'_z \frac{dz}{dt} \right) = \int_a^b \left[f(x(t), y(t), z(t)) \right]'_t \, dt =$$

$$= f(x(b), y(b), z(b)) - f(x(a), y(a), z(a)).$$

Therefore

$$\int_C \vec{F} \cdot \vec{T} \, ds = f(B) - f(A).$$

Line integral depends only on the points A and B and is therefore independent of the choice of the particular path C.

Theorem 8. *If*

$$P'_y = Q'_x$$

then

$$\int_C P \, dx + Q \, dy$$

is independent of path and vice versa.

Let us compute line integral in this case:

$$I = \int_C P(x, y) \, dx + Q(x, y) \, dy =$$

$$= \int_a^b [P(x(t), y(t)) x'(t) + Q(x(t), y(t)) y'(t)] dt =$$

$$= \int_a^b [U(x(t), y(t))]'_t \, dt = U(x(b), y(b)) - U(x(a), y(a)) =$$
$$= U(x_1, y_1) - U(x_0, y_0).$$

Then

$$U'_x(x, y) = P(x, y) \implies U(x, y) = \int_{x_0}^{x} P(x, y) \, dx + U(x_0, y),$$

$$U'_y(x, y) = Q(x, y) \implies U(x, y) = \int_{y_0}^{y} Q(x, y) \, dy + U(x, y_0)$$

and, consequently,

$$U(x, y) = \int_{x_0}^{x} P(x, y) \, dx + \int_{y_0}^{y} Q(x_0, y) \, dy + U(x_0, y_0).$$

Therefore

$$I = U(x_1, y_1) - U(x_0, y_0) = \int_{x_0}^{x_1} P(x, y_1) \, dx + \int_{y_0}^{y_1} Q(x_0, y) \, dy.$$

3.8 Green's Theorem

Let C be a piecewise smooth simple closed curve that bounds the region D in the plane. Suppose that the functions $P(x, y)$ and $Q(x, y)$ are continuous and have continuous first-order partial derivatives on D. Then

$$\int_{C^+} P(x, y) \, dx + Q(x, y) \, dy = \iint_D (Q'_x(x, y) - P'_y(x, y)) \, dxdy.$$

The positive or counterclockwise direction along C is the direction, determining by a parameterization $r(t)$ of C such that the region D remains on the left as the point $r(t)$ traces the boundary curve C. A reverse direction is negative, or the clockwise.

3.9 Corollary to Green's Theorem

The area A of the region D bounded by the piecewise smooth simple closed curve C is given by formula

$$A = \frac{1}{2} \int_{C^+} (-y \, dx + x \, dy) = -\int_{C^+} y \, dx = \int_{C^+} x \, dy.$$

Proof. With $P(x,y) \equiv -y, Q(x,y) \equiv 0$ Green's theorem gives

$$-\int_{C^+} y\,dx = \iint_D dx\,dy = A;$$

similarly with $P(x,y) \equiv 0, Q(x,y) \equiv x$ we have

$$\int_{C^+} x\,dy = \iint_D dx\,dy = A.$$

The sum of these results gives us the third formula.

Chapter 4

SURFACE INTEGRALS

4.1 Parametric Surface

A **parametric surface** S is the **image** of a function or transformation $\vec{r}$ that is defined on a region D in the uv-plane and has values in xyz-space. The **image** under $\vec{r}$ of each point (u, v) in D is the point in xyz-space with position vector

$$\vec{r}(u, v) = (x(u, v), (u, v), (u, v))$$

where $(u, v) \in D$. Let $u = C$ (i.e. u is a fixed constant C). Let us define the vector

$$\vec{S}_v = \vec{r}(C, v + \Delta v) - \vec{r}(C, v)$$

and, moreover, let us define a vector $\vec{T}_v$ as limit (provided its existence):

$$\vec{T}_v = \lim_{\Delta v \to 0} \frac{\vec{S}_v}{\Delta v} \; .$$

Then

$$\vec{T}_v =$$

$$\lim_{\Delta v \to 0} \frac{1}{\Delta v}[(x(C, v+\Delta v)-x(C, v), y(C, v+\Delta v)-y(C, v), z(C, v+\Delta v)-z(C, v))]$$

and, consequently,

$$\vec{T}_v = (x'_v(C, v), y'_v(C, v), z'_v(C, v)) \equiv \vec{r}_v.$$

Therefore

$$\vec{T}_v = \vec{r}_v = (x'_v, y'_v, z'_v)$$

and, by analogy,

$$\vec{T}_u = \vec{r}_u = (x'_u, y'_u, z'_u).$$

We call the variables u and v the **parameters** for the surface S

Example 11. We may regard the graph

$$z = f(x, y)$$

of a function as a parametric surface with parameters x and y. In this case the transformation $\vec{r}$ from the xy-plane to xyz-space has the component functions

$$x = x \quad y = y \quad z = f(x, y).$$

4.2 Surface Area

Now we want **to define the surface area of the parametric surface**. We begin with an inner partition of D into rectangles $D_1, D_2, \dots, D_n$, each with dimensions Δu and Δv. Let (u_i, v_i) be the lower left-hand corner of D_i. The image S_i of D_i under $\vec{r}$ will not generally be a rectangle in $xyz-$space. It will look more like a **curvilinear figure** on the image surface S with $\vec{r}(u_i, v_i)$ as one **vertex.**

Let ΔS_i denote the area of this curvilinear figure S_i. The parametric curves $\vec{r}(u, v_i)$ and $\vec{r}(u_i, v)$ - with parameters u and v, respectively, - lie on the surface S and met at the point $\vec{r}(u_i, v_i)$. At this point of intersection, these two curves have the tangent vectors $\vec{r}_u(u_i, v_i)$ and $\vec{r}_v(u_i, v_i)$. Hence their **vector product**

$$\vec{N}(u_i, v_i) = \vec{r}_u(u_i, v_i) \times \vec{r}_v(u_i, v_i)$$

is a **normal vector** to S at the point $\vec{r}(u_i, v_i)$. Now suppose that both Δu and Δv are small. Then the area $\Delta S\ _i$ of the curvilinear figure S_i will be approximatively equal to the area ΔP_i of the **adjacent parallelogram** with sides $\vec{r}_u(u_i, v_i)\Delta u$ and $\vec{r}_v(u_i, v_i)\Delta v$ (because $\vec{S}_v \approx \vec{T}_v\Delta v\ = \vec{r}_v\Delta v$ and by analogy $\vec{S}_u \approx \vec{T}_u\Delta u\ = \vec{r}_u\Delta u$). But the area of this parallelogram is:

$$\Delta P_i = |\vec{r}_v\Delta v \times \vec{r}_u\Delta u\ | = |\vec{r}_u \times \vec{r}_v|\Delta u\ \Delta v\ = |\vec{N}(u_i, v_i)| \cdot \Delta u\ \Delta v\ .$$

This means that the area $a(S)$ of the surface S is given approximatively by

$$a(S) = \sum_{i=1}^{n} \Delta S_i \approx \sum_{i=1}^{n} \Delta P_i = \sum_{i=1}^{n} |\vec{N}(u_i, v_i)| \cdot \Delta u\ \Delta v.$$

This last sum is a Riemann sum for the double integral

$$\iint_D |\vec{N}(u,v)|\, dudv.$$

We are therefore motivated to define the surface area A of the surface A of the parametric surface S by

$$A = a(S) = \iint_D |\vec{N}(u,v)|\, dudv = \iint_D |\vec{r}_u \times \vec{r}_v|\, dudv.$$

4.3 Surface Area in Rectangular Coordinates

In the case of the surface $z = f(x, y)$, $(x, y) \in D$ we will put $u = x$ and $v = y$. Then

$$\vec{T}_x = (1, 0, f'_x(x, y)),$$
$$\vec{T}_y = (0, 1, f'_y(x, y))$$

and

$$\vec{N} = \vec{T}_x \times \vec{T}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f'_x(x,y) \\ 0 & 1 & f'_y(x,y) \end{vmatrix} = -f'_x(x,y)\vec{i} - f'_y(x,y)\vec{j} + \vec{k}$$

and

$$A = a(S) = \iint_D \sqrt{1 + (f'_x(x,y))^2 + (f'_y(x,y))^2}\, dxdy.$$

Example 12. Let us find the area of the ellipse cut from the plane $z = 2x + 2y + 1$ by the cylinder $x^2 + y^2 = 1$.
Solution. By formula above we have

$$A = \iint_D \sqrt{1+4+4}\, dxdy = \iint_D 3\, dxdy = 3 \iint_D dxdy = 3\pi.$$

Example 13. Let a domain $\mathcal{D}$ having the area $\mathcal{A}$, lying in a plane ρ:

$$Ax + By + Cz + D = 0$$

is given. Find the area $\mathcal{A}'$ of the domain $\mathcal{D}'$ which is the perpendicular projection of $\mathcal{D}$ to the plane xy.

Solution. Let us suppose that the unit normal vector of the plane ρ is $\vec{n} = (n_x, n_y, n_z) = (\cos\alpha, \cos\beta, \cos\gamma)$ and that $\gamma \in [0, \pi/2]$. Then $A = \cos\alpha$, $B = \cos\beta$ and $C = \cos\gamma$, i.e. the equation of ρ can be rewritten as

$$\cos\alpha \cdot x + \cos\beta \cdot y + \cos\gamma \cdot z + D = 0.$$

Excluding the trivial case when $\gamma = \pi/2$, i.e. $\cos\gamma = 0$, when the area of projection equals zero we get

$$z = \frac{D - \cos\alpha \cdot x - \cos\beta \cdot y}{\cos\gamma}.$$

Then

$$\mathcal{A} = \iint_{\mathcal{D}'} \sqrt{1 + (z'_x)^2 + (z'_y)^2}\mathrm{d}x\mathrm{d}y =$$

$$\iint_{\mathcal{D}'} \sqrt{1 + \left(-\frac{\cos\alpha}{\cos\gamma}\right)^2 + \left(-\frac{\cos\beta}{\cos\gamma}\right)^2}\mathrm{d}x\mathrm{d}y =$$

$$\iint_{\mathcal{D}'} \sqrt{\frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}{\cos\gamma}}\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{D}'} \sqrt{\frac{\|\vec{n}\|}{\cos\gamma}}\mathrm{d}x\mathrm{d}y =$$

$$\frac{1}{\cos\gamma}\iint_{\mathcal{D}'} \mathrm{d}x\mathrm{d}y = \frac{1}{\cos\gamma} \cdot \mathcal{A}'.$$

Then the area $\mathcal{A}'$ of the domain $\mathcal{D}'$ is given by relation

$$\mathcal{A}' = \mathcal{A} \cdot \cos\gamma.$$

This formula is valid for all values $\gamma \in [0, \pi/2]$.

4.4 Surface Integrals - First Type of Surface Integral

Definition 8. The surface integral of the first type of the function $f(x, y, z)$ over the surface S is defined as:

$$\iint_S f(x, y, z)\, dS = \iint_D f(\vec{r}(u, v))|\vec{N}(u, v)|\, dudv =$$

$$\lim_{n,m\to\infty,\Delta\to 0} \sum_{i=1,j=1}^{n,m} f(\vec{r}(u_i,v_j))|\vec{N}(u_i,v_j)|\,\Delta u_i\ \Delta v_j.$$

(Let us note that $\vec{N}(u,v) = \vec{r}_u \times \vec{r}_v$.)
If the surface S is described by means of equation $z = h(x,y)$ for $(x,y) \in D \subset \mathbb{R}^2$ we may use x and y as the parameters (rather than u and v). In this case we put $u = x, v = y$,

$$\vec{r}_u = \vec{r}_x = (1, 0, h'_x(x,y)),$$

$$\vec{r}_v = \vec{r}_y = (0, 1, h'_y(x,y)),$$

$$|\vec{r}_x \times \vec{r}_y| = \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & h'_x(x,y) \\ 0 & 1 & h'_y(x,y) \end{matrix} \right\| = |-\vec{i}h'_x(x,y) - \vec{j}h'_y(x,y) + \vec{k}| =$$

$$= \sqrt{1 + (h'_x(x,y))^2 + (h'_y(x,y))^2}$$

and

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,h(x,y))\sqrt{1 + (h'_x(x,y))^2 + (h'_y(x,y))^2}\,dxdy.$$

4.5 Surface Integrals - Second Type of Surface Integral

The surface integral $\iint_S f(x,y,z)\,dS$ is analogous to the line integral $\int_C f(x,y)\,ds$. There is a second type of surface integral that is analogous to the line integral of the form $\int_C P\,dx + D\,dy$. To define the surface integral

$$\iint_S f(x,y,z)\,dxdy$$

with $dxdy$ in place of dS, we replace the area ΔP_i in definition of the surface integral of the first type by the area of its projection into the $xy-$plane with corresponding sign. To see how this works out, consider the unit normal vector to S,

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|} = \vec{i}\cos\alpha + \vec{j}\cos\beta + \vec{k}\cos\gamma.$$

Because

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{vmatrix}$$

i.e.

$$\vec{N} = \vec{i}\,\frac{\partial(y,z)}{\partial(u,v)} + \vec{j}\,\frac{\partial(z,x)}{\partial(u,v)} + \vec{k}\,\frac{\partial(x,y)}{\partial(u,v)},$$

the components of the unit normal vector $\vec{n}$ are

$$\cos\alpha = \frac{1}{|\vec{N}|}\,\frac{\partial(y,z)}{\partial(u,v)},\ \cos\beta = \frac{1}{|\vec{N}|}\,\frac{\partial(z,x)}{\partial(u,v)},\ \cos\gamma = \frac{1}{|\vec{N}|}\,\frac{\partial(x,y)}{\partial(u,v)}\ .$$

Then

$$\iint_S f(x,y,z)\,dxdy = \iint_S f(x,y,z)\cos\gamma\,dS = \iint_D f(\vec{r}(u,v))\frac{\partial(x,y)}{\partial(u,v)}\,dudv$$

where

$$dS = |\vec{N}(u,v)|dudv = |\vec{r}'_u \times \vec{r}'_v|dudv.$$

Similarly we define

$$\iint_S f(x,y,z)\,dydz = \iint_S f(x,y,z)\cos\alpha\,dS = \iint_S f(\vec{r}(u,v))\frac{\partial(y,z)}{\partial(u,v)}\,dudv$$

and

$$\iint_S f(x,y,z)\,dzdx = \iint_S f(x,y,z)\cos\beta\,dS = \iint_S f(\vec{r}(u,v))\frac{\partial(z,x)}{\partial(u,v)}\,dudv.$$

The general surface integral of the second type is defined as the sum

$$\iint_S P(x,y,z)\,dydz + Q(x,y,z)\,dzdx + R(x,y,z)\,dxdy =$$
$$= \iint_S (P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma)\,dS =$$
$$= \iint_D \left(P(\vec{r}(u,v))\frac{\partial(y,z)}{\partial(u,v)} + Q(\vec{r}(u,v))\frac{\partial(z,x)}{\partial(u,v)} + R(\vec{r}(u,v))\frac{\partial(x,y)}{\partial(u,v)}\right)dudv.$$

Here $P(x,y,z), Q(x,y,z)$ and $R(x,y,z)$ are continuous functions of x, y and z.

Let us suppose that S is the surface $z = h(x, y), (x, y) \in D$. Then we can put $u = x, y = v$ and

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial(y,z)}{\partial(x,y)} = \begin{vmatrix} 0 & z'_x(x,y) \\ 1 & z'_y(x,y) \end{vmatrix} = -h'_x(x,y),$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \frac{\partial(z,x)}{\partial(x,y)} = \begin{vmatrix} z'_x(x,y) & 1 \\ z'_y(x,y) & 0 \end{vmatrix} = -h'_y(x,y),$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Hence

$$\iint_S P(x,y,z)\,dydz + Q(x,y,z)\,dzdx + R(x,y,z)\,dxdy =$$
$$= \iint_D (-P(x,y,h(x,y))h'_x - Q(x,y,h(x,y)))h'_y + R(x,y,h(x,y)))) \,dxdy.$$

4.6 The Divergence Theorem

(The Gauss'-Ostrogradski's Theorem)

Let us suppose that S is closed piecewise smooth surface that bounds the space region ω. Let

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

be a vector field with component functions that have continuous first-order partial derivatives on ω. Let $\vec{n}$ be the **outer** unit normal vector to S. Then the **divergence theorem** (or so called **Gauss'-Ostrogradski's theorem**) is expressed by means of formula

$$\iint_S \vec{F} \cdot \vec{n}\,dS = \iiint_\omega \vec{\nabla} \cdot \vec{F}\,dV. \tag{4.1}$$

Let us denote $dV = dx\,dy\,dz$,

$$\vec{\nabla} = \frac{\partial}{\partial x} \cdot \vec{i} + \frac{\partial}{\partial y} \cdot \vec{j} + \frac{\partial}{\partial z} \cdot \vec{k}$$

and define so called **divergence** of the vector field $\vec{F}$:

$$\mathrm{div}\vec{F}(x,y,z) = \vec{\nabla} \cdot \vec{F}(x,y,z) = P'_x(x,y,z) + Q'_y(x,y,z) + R'_z(x,y,z).$$

Let us express the outer unit vector $\vec{n}$ as

$$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$$

Then the divergence theorem (4.1) can be rewritten in the form

$$\iint_S [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma]\, dS =$$
$$= \iiint_\omega [P'_x(x,y,z) + Q'_y(x,y,z) + R'_z(x,y,z)]\, dxdydz,$$

or in the form

$$\iint_S P(x,y,z)\, dydz + Q(x,y,z)\, dzdx + R(x,y,z)\, dxdy =$$
$$= \iiint_\omega [P'_x(x,y,z) + Q'_y(x,y,z) + R'_z(x,y,z)]\, dxdydz.$$

4.7 Stoke's Theorem

Definition 9. An oriented surface is a surface together with a chosen continuous unit normal vector field $\vec{n}$. The **positive orientation** of the boundary C of an oriented surface S corresponds to the unit tangent vector $\vec{T}$ such that $\vec{n} \times \vec{T}$ always points into S. Check that for a **plane region** with unit normal vector $\vec{k}$, the positive orientation of its outer boundary is counterclockwise.

Theorem 9 (Stoke's Theorem) *Let S be an oriented, bounded, and piecewise smooth surface in space with positively oriented boundary. Suppose that the components of the vector field $\vec{F}(x,y,z)$ have continuous first-order partial derivatives in a space region that contains S. Then*

$$\oint_{L^+} \vec{F} \cdot \vec{T}\, ds = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS \tag{4.2}$$

where L is boundary curve of S.

Note that $\operatorname{curl} \vec{F}$ (or $\operatorname{rot} \vec{F}$) is so called **rotation** of the vector $\vec{F}$ and is computed by formula

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} =$$
$$= (R'_y - Q'_z)\vec{i} + (P'_z - R'_x)\vec{j} + (Q'_x - P'_y)\vec{k}.$$

Since we can express

$$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$$

and

$$\vec{T}\,ds = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz,$$
$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

then the formula (4.2) can be rewritten in the form

$$\int_{L^+} P\,dx + Q\,dy + R\,dz =$$
$$\iint_S (R'_y - Q'_z)\,dydz + (P'_z - R'_x)\,dzdx + (Q'_x - P'_y)\,dxdy.$$

Index

Contents