TAYLORŮV POLYNOM

Taylorův polynom stupně n funkce f v bodě x_0:

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Je-li $x_0 = 0$, pak se T_n nazývá Maclaurinův polynom.

Pozn.: V okolí bodu x_0 Taylorův polynom přibližně nahrazuje funkci f; tedy platí:

$$f(x) \doteq T_n(x)$$

Př.: Určete Taylorův (Maclaurinův) polynom stupně n funkce f v bodě x_0.

1) $f(x) = \sqrt{x+1}$, $x_0 = 0$, $n = 3$

$$f(x) = (x+1)^{\frac{1}{2}} \qquad f(0) = \sqrt{0+1} = 1$$

$$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+1}} \qquad f'(0) = \frac{1}{2\sqrt{0+1}} = \frac{1}{2}$$

$$f''(x) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)(x+1)^{-\frac{3}{2}} = -\frac{1}{4\sqrt{(x+1)^3}} \qquad f''(0) = -\frac{1}{4\sqrt{(0+1)^3}} = -\frac{1}{4}$$

$$f'''(x) = -\frac{1}{4}\cdot\left(-\frac{3}{2}\right)(x+1)^{-\frac{5}{2}} = \frac{3}{8\sqrt{(x+1)^5}} \qquad f'''(0) = \frac{3}{8\sqrt{(0+1)^5}} = \frac{3}{8}$$

Maclaurinův polynom 3. stupně:

$$T_3(x) = f(0) + \frac{f'(0)}{1!}\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \frac{f'''(0)}{3!}\cdot x^3 =$$

$$= 1 + \frac{\frac{1}{2}}{\underbrace{1!}_{=1}}\cdot x + \frac{-\frac{1}{4}}{\underbrace{2!}_{=2\cdot 1}}\cdot x^2 + \frac{\frac{3}{8}}{\underbrace{3!}_{=3\cdot 2\cdot 1}}\cdot x^3 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

2) $f(x) = x\cdot \operatorname{arctg} x\,, \quad x_0 = 1\,, \quad n = 2$

$$f'(x) = 1\cdot \operatorname{arctg} x + x\cdot \frac{1}{1+x^2} = \operatorname{arctg} x + \frac{x}{1+x^2}$$

$$f''(x) = \frac{1}{1+x^2} + \frac{1+x^2 - x\cdot 2x}{(1+x^2)^2} = \frac{1+x^2+1+x^2-2x^2}{(1+x^2)^2} = \frac{2}{(1+x^2)^2}$$

$$f(1) = 1\cdot \underbrace{\operatorname{arctg} 1}_{\frac{\pi}{4}} = \frac{\pi}{4}$$

Pozn.: $\operatorname{arctg} 1 = \alpha \Rightarrow \operatorname{tg}\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$

$$f'(1) = \operatorname{arctg} 1 + \frac{1}{1+1^2} = \frac{\pi}{4} + \frac{1}{2}$$

$$f''(1) = \frac{2}{(1+1^2)^2} = \frac{2}{2^2} = \frac{1}{2}$$

$$T_2(x) = f(x_0) + \frac{f'(x_0)}{1!}\cdot(x-x_0) + \frac{f''(x_0)}{2!}\cdot(x-x_0)^2 =$$

$$= \frac{\pi}{4} + \frac{\frac{\pi}{4}+\frac{1}{2}}{1!}\cdot(x-1) + \frac{\frac{1}{2}}{2!}\cdot(x-1)^2 = \frac{\pi}{4} + \left(\frac{\pi}{4}+\frac{1}{2}\right)\cdot(x-1) + \frac{1}{4}(x-1)^2$$

$$= \frac{\pi}{4} + \frac{\pi+2}{4}(x-1) + \frac{1}{4}(x-1)^2$$

3) $f(x) = \frac{1}{x}\,, \quad x_0 = 2\,, \quad n = 4$

$$f(x) = x^{-1}$$

$$f'(x) = -x^{-2} = -\frac{1}{x^2}$$

$$f''(x) = -(-2)\cdot x^{-3} = 2x^{-3} = \frac{2}{x^3}$$

$$f'''(x) = 2\cdot(-3)\cdot x^{-4} = -6x^{-4} = -\frac{6}{x^4}$$

$$f^{(4)}(x) = -6\cdot(-4)\cdot x^{-5} = 24x^{-5} = \frac{24}{x^5}$$

$$f(2) = \frac{1}{2}$$

$$f'(2) = -\frac{1}{2^2} = -\frac{1}{4}$$

$$f''(2) = \frac{2}{2^3} = \frac{1}{2^2} = \frac{1}{4}$$

$$f'''(2) = -\frac{6}{2^4} = -\frac{3}{2^3} = -\frac{3}{8}$$

$$f^{(4)}(2) = \frac{24}{2^5} = \frac{2^3\cdot 3}{2^5} = \frac{3}{2^2} = \frac{3}{4}$$

$$T_4(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4 =$$

$$= \frac{1}{2} + \frac{-\frac{1}{4}}{\underbrace{1!}_{=1}}(x-2) + \frac{\frac{1}{4}}{\underbrace{2!}_{=2\cdot 1}}(x-2)^2 + \frac{-\frac{3}{8}}{\underbrace{3!}_{=3\cdot 2\cdot 1}}(x-2)^3 + \frac{\frac{3}{4}}{\underbrace{4!}_{=4\cdot 3\cdot 2\cdot 1}}(x-2)^4 =$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4$$