Příklad 1.1.1. Vypočtěte matici $D = -\frac{1}{2}A + 3B$, jestliže:

$$A = \begin{pmatrix} 2 & 1 & 4 \\ -2 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 2 & 6 \\ 0 & 3 & 6 \end{pmatrix}.$$

2/3 2/3

$$D = -\frac{1}{2}\begin{pmatrix} 2 & 1 & 4 \\ -2 & 0 & 6 \end{pmatrix} + 3\begin{pmatrix} -4 & 2 & 6 \\ 0 & 3 & 6 \end{pmatrix} = \begin{pmatrix} -1 & -\frac{1}{2} & -2 \\ 1 & 0 & -3 \end{pmatrix} + \begin{pmatrix} -12 & 6 & 18 \\ 0 & 9 & 18 \end{pmatrix} = \begin{pmatrix} -13 & \frac{11}{2} & 16 \\ 1 & 9 & 15 \end{pmatrix}$$

Příklad 1.1.2. Vynásobte matice

$$\underset{2/3}{\overset{A}{\begin{pmatrix} 2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}}} \cdot \underset{3/2}{\overset{B}{\begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}}}. \qquad C_{2/2}$$

$$A_{m,p} \cdot B_{p,n} = C_{m,n}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ 10 & 3 \end{pmatrix}$$

$$A \cdot B = C_{2/2}$$

$$B_{3/2} \cdot A_{2/3} = D_{3/3}$$

Příklad 1.1.3. Vypočtěte $B = A^2 - A - E^3$, kde $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$A^2 = A \cdot A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 4 \\ 11 & 2 & 5 \\ -1 & 0 & -1 \end{pmatrix} \qquad E^3 = E \cdot E \cdot E = E^2 \cdot E = E \cdot E^2 = E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 8 & 2 & 4 \\ 11 & 2 & 5 \\ -1 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 3 \\ 8 & 0 & 3 \\ -2 & 1 & -2 \end{pmatrix}$$

Příklad 1.2.1. Určete hodnost matice

1.
$$\begin{pmatrix} 4 & 8 & 4 & 4 & 8 \\ 3 & 2 & 7 & -5 & -6 \\ 3 & 6 & 3 & 3 & 6 \\ -5 & -7 & -8 & 1 & -1 \end{pmatrix} \sim \begin{array}{c} \begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 3 & 2 & 7 & -5 & -6 \\ 1 & 2 & 1 & 1 & 2 \\ -5 & -7 & -8 & 1 & -1 \end{pmatrix} \end{array} \begin{array}{l} -1 \; -3 \; 5 \\ \\ \\ \\ \end{array} \sim \begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & -4 & 4 & -8 & -12 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 3 & -3 & 6 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & 2 & 3 \\ \cancel{0} & \cancel{1} & \cancel{-1} & \cancel{2} & \cancel{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ 1 \\ + \\ \\ \end{array} \sim \begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$h(A) = 2$

$A =$ **2.** $\begin{pmatrix} 2 & 1 & 4 & 5 \\ 4 & 3 & 5 & 6 \\ 1 & -1 & 0 & 15 \\ 3 & 0 & 1 & 4 \\ 0 & 2 & 7 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 15 \\ 2 & 1 & 4 & 5 \\ 4 & 3 & 5 & 6 \\ 3 & 0 & 1 & 4 \\ 0 & 2 & 7 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 15 \\ 0 & 3 & 4 & -25 \\ 0 & 7 & 5 & -54 \\ 0 & 3 & 1 & -41 \\ 0 & 2 & 7 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 15 \\ 0 & 3 & 4 & -25 \\ 0 & 0 & -13 & 13 \\ 0 & 0 & -3 & -16 \\ 0 & 0 & 13 & 65 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 15 \\ 0 & 3 & 4 & -25 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -3 & -16 \\ 0 & 0 & 1 & 5 \end{pmatrix}$

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$\sim \begin{pmatrix} 1 & -1 & 0 & 15 \\ 0 & 3 & 4 & -25 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -19 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{6} \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 15 \\ 0 & 3 & 4 & -25 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$h(A) = 4$

$A^T = \begin{pmatrix} 2 & 4 & 1 & 3 & 0 \\ 1 & 3 & -1 & 0 & 2 \\ 4 & 5 & 0 & 1 & 7 \\ 5 & 6 & 15 & 4 & 5 \end{pmatrix} \sim \ldots$

Příklad 1.3.1. Gaussovou eliminační metodou řešte systémy lineárních algebraických rovnic nad tělesem $\mathbb{R}$. Vždy proveďte rozbor řešitelnosti a počtu řešení na základě Frobeniovy věty.

1.
$$\begin{aligned} 2x_1 &+ x_2 &+ x_3 &= 2 \\ x_1 &+ 3x_2 &+ x_3 &= 5 \\ x_1 &+ x_2 &+ 5x_3 &= -7 \\ 2x_1 &+ 3x_2 &- 3x_3 &= 14 \end{aligned}$$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \\ 2 & 3 & -3 \end{pmatrix} ; \quad A_R = \left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 1 & 3 & 1 & 5 \\ 1 & 1 & 5 & -7 \\ 2 & 3 & -3 & 14 \end{array}\right)$$

1. $h(A) \neq h(A_R) \Rightarrow \nexists$ řešení
2. $h(A) = h(A_R) \Rightarrow \exists$ řešení

a) $h(A) = h(A_R) = n \Rightarrow$ má právě jedno řešení

b) $h(A) = h(A_R) < n \Rightarrow \infty$ řešení

$n - h$ ... počet volných parametrů

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \downarrow & \downarrow & \downarrow & \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 1 & 3 & 1 & 5 \\ 1 & 1 & 5 & -7 \\ 2 & 3 & -3 & 14 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & -7 \\ 1 & 3 & 1 & 5 \\ 2 & 1 & 1 & 2 \\ 2 & 3 & -3 & 14 \end{array}\right) \begin{matrix} \cdot(-1)\ \cdot(-2) \\ +\\ + \\ + \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & -7 \\ 0 & 2 & -4 & 12 \\ 0 & -1 & -9 & 16 \\ 0 & 1 & -13 & 28 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & -7 \\ 0 & 1 & -2 & 6 \\ 0 & -1 & -9 & 16 \\ 0 & 1 & -13 & 28 \end{array}\right) \begin{matrix} \\ \cdot(-1) \\ + \\ + \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & -7 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & -11 & 22 \\ \cancel{0} & \cancel{0} & \cancel{-11} & \cancel{22} \end{array}\right)$$

$h(A) = 3 = h(A_R) \Rightarrow \exists!$ řešení

$-11x_3 = 22 \Rightarrow x_3 = -2$

$x_2 - 2x_3 = 6$

$x_2 - 2(-2) = 6 \Rightarrow x_2 = 2$

$x_1 + x_2 + 5x_3 = -7$

$x_1 + 2 + 5(-2) = -7 \Rightarrow x_1 = 1$

$K = \{(1, 2, -2)\}$

$$\textbf{2.}\quad \begin{aligned} x - 2y - 5z &= 2\\ 2x + 3y - z &= -1\\ -8x - 19y - 5z &= 7 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & -5 & 2 \\ 2 & 3 & -1 & -1 \\ -8 & -19 & -5 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -5 & 2 \\ 0 & 7 & 9 & -5 \\ 0 & -35 & -45 & 23 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & -5 & 2 \\ 0 & 7 & 9 & -5 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

$h(A) = 2 \neq h(A_r) = 3 \Rightarrow \nexists$ řešení

$$\textbf{4.}\quad \begin{aligned} 7x_1 + 3x_2 - x_3 + 2x_4 &= 2 \\ x_1 - 2x_2 + 3x_3 - x_4 &= -3 \\ 6x_1 + 5x_2 - 4x_3 + 3x_4 &= 5 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & -3 \\ 7 & 3 & -1 & 2 & 2 \\ 6 & 5 & -4 & 3 & 5 \end{array}\right) \begin{array}{c} -7 \;\; -6 \\ \\ \\ \end{array} \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & -3 \\ 0 & 17 & -22 & 9 & 23 \\ \cancel{0} & \cancel{17} & \cancel{-22} & \cancel{9} & \cancel{23} \end{array}\right)$$

$h(A) = 2 = h(A_r) < 4 = n$

volíme $4-2=2$ voľných neznámych

$x_2 = t$

$x_3 = s$

$17x_2 - 22x_3 + 9x_4 = 23$

$17t - 22s + 9x_4 = 23$

$9x_4 = 23 - 17t + 22s$

$x_4 = \frac{23}{9} - \frac{17}{9}t + \frac{22}{9}s$

$x_1 - 2x_2 + 3x_3 - x_4 = -3$

$x_1 - 2t + 3s - \left(\frac{23}{9} - \frac{17}{9}t + \frac{22}{9}s\right) = -3$

$x_1 = -\frac{4}{9} + \frac{1}{9}t - \frac{5}{9}s$

$$K = \left\{\left(-\frac{4}{9} + \frac{1}{9}t - \frac{5}{9}s,\ t,\ s,\ \frac{23}{9} - \frac{17}{9}t + \frac{22}{9}s\right),\ t, s \in \mathbb{R}\right\}$$

$$\begin{array}{rcrcrcl} \alpha x &+& y &+& z &=& 1 \\ x &+& \alpha y &+& z &=& \alpha \\ x &+& y &+& \alpha z &=& \alpha^2 \end{array}$$

6.

$$\left(\begin{array}{ccc|c} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & \alpha \\ 1 & 1 & \alpha & \alpha^2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & \alpha & \alpha^2 \\ 1 & \alpha & 1 & \alpha \\ \alpha & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & \alpha & \alpha^2 \\ 0 & \alpha-1 & 1-\alpha & \alpha-\alpha^2 \\ 0 & 1-\alpha & 1-\alpha^2 & 1-\alpha^3 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & \alpha & \alpha^2 \\ 0 & -(1-\alpha) & 1-\alpha & \alpha(1-\alpha) \\ 0 & 1-\alpha & (1-\alpha)(1+\alpha) & (1-\alpha)(1+\alpha+\alpha^2) \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & \alpha & \alpha^2 \\ 0 & -(1-\alpha) & 1-\alpha & \alpha(1-\alpha) \\ 0 & 0 & (1-\alpha)(2+\alpha) & (1-\alpha)(1+\alpha)^2 \end{array}\right)$$

$(1-\alpha)+(1-\alpha)(1+\alpha) = (1-\alpha)(1+1+\alpha) = (1-\alpha)(2+\alpha)$

$\alpha(1-\alpha)+(1-\alpha)(1+\alpha+\alpha^2) = (1-\alpha)\cdot(\alpha+1+\alpha+\alpha^2) = (1-\alpha)\cdot(1+2\alpha+\alpha^2) = (1-\alpha)(1+\alpha)^2$

$\alpha_1 = 1$: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $h(A) = 1 = h(A_r) < 3$ $3-1$ volných neznámých

$K = \{(1-s-t, s, t);\ s, t \in \mathbb{R}\}$

$\alpha_2 = -2$: $\left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & -3 & 3 & -6 \\ 0 & 0 & 0 & 3 \end{array}\right)$ $h(A) = 2 \neq h(A_r) = 3 \Rightarrow \nexists$ řešení

$\alpha \neq 1 \wedge \alpha \neq -2$ $(1-\alpha)(2+\alpha)x_3 = (1-\alpha)(1+\alpha)^2$

$$x_3 = \frac{\cancel{(1-\alpha)}(1+\alpha)^2}{\cancel{(1-\alpha)}(2+\alpha)} = \frac{(1+\alpha)^2}{2+\alpha}$$

$\vdots$

$$\left(-\frac{\alpha+1}{\alpha+2},\ \frac{1}{\alpha+2},\ \frac{(1+\alpha)^2}{2+\alpha}\right)$$

$\Uparrow x_1$ $\Uparrow x_2$ $\Uparrow x_3$

$h(A) = h(A_r) = 3 \Rightarrow$ jediné řešení

$$
\textbf{7.}\quad
\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & + & x_3 & = & 0 \\
3x_1 & - & 3x_2 & - & 2x_3 & = & -3 \\
7x_1 & - & 3x_2 & + & x_3 & = & 16
\end{array}
$$

$$
\left(\begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
3 & -3 & -2 & -3 \\
7 & -3 & 1 & 16
\end{array}\right)
\begin{array}{l} -3 \;\; -7 \\ \\ \\ \end{array}
\sim \cdots \sim
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
0 & 1 & -5 & -3 \\
0 & 0 & \cancel{49}\ 1 & \cancel{49}\ 1
\end{array}\right)
\begin{array}{l} \\ 5 \\ \\ \end{array}
\sim
$$

$x_3 = 1$

$x_2 - 5x_3 = -3 \qquad x = \ldots$

$x_1 = \ldots$

$$
\sim
\left(\begin{array}{ccc|c}
1 & -2 & 0 & -1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1
\end{array}\right)
\begin{array}{l} + \\ 2 \\ \\ \end{array}
\sim
\left(\begin{array}{ccc|c}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1
\end{array}\right)
\implies
\begin{array}{l}
x_1 = 3 \\
x_2 = 2 \\
x_3 = 1
\end{array}
$$