3.

1. Zjednodušte výrazy:

   a) $(\alpha-\beta)(\vec{a}+\vec{b})-(\alpha+\beta)(\vec{a}-\vec{b})$,

   b) $(\operatorname{tg}\frac{\pi}{4})\vec{a}-(\operatorname{cotg}\frac{\pi}{4})\vec{a}$,

   c) $(\alpha-\beta)^2(\vec{a}+\vec{b})-(\alpha+\beta)^2(\vec{a}-\vec{b})$,

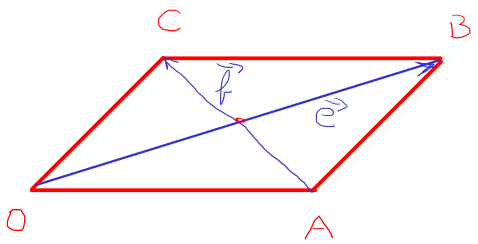
   kde $\alpha,\beta\in\mathbb{R}$; $\vec{a},\vec{b}\in\mathbb{V}(\mathbb{E}_3)$.

a) $(\alpha-\beta)(\vec{a}+\vec{b})-(\alpha+\beta)(\vec{a}-\vec{b})=\cancel{\alpha\vec{a}}+\alpha\vec{b}-\beta\vec{a}-\cancel{\beta\vec{b}}-\cancel{\alpha\vec{a}}+\alpha\vec{b}-\beta\vec{a}+\cancel{\beta\vec{b}}=2\alpha\vec{b}-2\beta\vec{a}=$
$=2(\alpha\vec{b}-\beta\vec{a})$

b) $(\operatorname{tg}\frac{\pi}{4})\vec{a}-(\operatorname{cotg}\frac{\pi}{4})\vec{a}=1\cdot\vec{a}-1\cdot\vec{a}=\vec{o}$

c) $(\alpha-\beta)^2(\vec{a}+\vec{b})-(\alpha+\beta)^2(\vec{a}-\vec{b})=(\alpha^2-2\alpha\beta+\beta^2)(\vec{a}+\vec{b})-(\alpha^2+2\alpha\beta+\beta^2)(\vec{a}-\vec{b})=$
$=\cancel{\alpha^2\vec{a}}-2\alpha\beta\vec{a}+\cancel{\beta^2\vec{a}}+\alpha^2\vec{b}-\cancel{2\alpha\beta\vec{b}}+\beta^2\vec{b}-\cancel{\alpha^2\vec{a}}-2\alpha\beta\vec{a}-\cancel{\beta^2\vec{a}}+\alpha^2\vec{b}+\cancel{2\alpha\beta\vec{b}}+\beta^2\vec{b}=$
$=-4\alpha\beta\vec{a}+2\alpha^2\vec{b}+2\beta^2\vec{b}=-4\alpha\beta\vec{a}+(2\alpha^2+2\beta^2)\vec{b}$

3. Dokažte vektorovou metodou, že čtyřúhelník $OABC$ je rovnoběžníkem právě tehdy, když se úhlopříčky čtyřúhelníku vzájemně půlí.

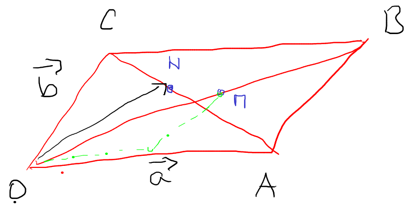


$A \Leftrightarrow B$ ekvivalence

$A \Rightarrow B$ implikace

a) „$\Rightarrow$" nutná podmínka

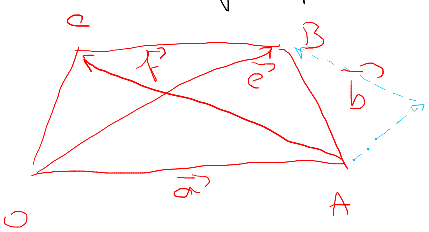
Důkaz sporem: předpokládám, že M, N jsou různé



$$\left.\begin{aligned} \vec{ON} &= \vec{a} + \frac{\vec{f}}{2} \\ \vec{a} + \vec{f} &= \vec{b} \Rightarrow \vec{f} = \vec{b} - \vec{a} \end{aligned}\right\} \Rightarrow$$

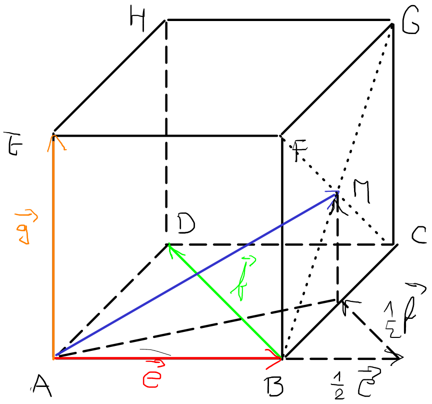
$$\Rightarrow \left.\begin{aligned} \vec{ON} &= \vec{a} + \frac{1}{2}(\vec{b} - \vec{a}) = \vec{a} + \frac{1}{2}\vec{b} - \frac{1}{2}\vec{a} = \frac{1}{2}(\vec{a} + \vec{b}) \\ \vec{OM} &= \frac{1}{2}(\vec{a} + \vec{b}) \end{aligned}\right\} \vec{OM} = \vec{ON} \Rightarrow M = N \Rightarrow \text{spor!}$$

b) „$\Leftarrow$" postačující podmínka



$$\left.\begin{aligned} \vec{AB} = \vec{b} &= \frac{1}{2}(\vec{e} + \vec{f}) \\ \vec{OC} &= \frac{1}{2}(\vec{e} + \vec{f}) \end{aligned}\right\} \vec{OC} = \vec{b}$$

6. Je dána krychle $ABCDEFGH$ a střed M stěny $BCGF$. Určete rozklad vektoru $\vec{u} = \overrightarrow{AM}$ do trojice vektorů $\overrightarrow{AB} = \vec{e}$, $\overrightarrow{BD} = \vec{f}$, $\overrightarrow{AE} = \vec{g}$.



$$\vec{u} = \frac{3}{2}\vec{e} + \frac{1}{2}\vec{f} + \frac{1}{2}\vec{g}$$

6. **Příklad 1.** Najděte vlastní čísla a vlastní vektory:

a) $\begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = A \qquad |A-\lambda E| = 0 \qquad (A-\lambda E)X = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} \overset{+\%}{=} (2-\lambda)^3 - \cancel{1} + \cancel{1} - (\cancel{2-\lambda}) + (\cancel{2-\lambda}) - (2-\lambda) = (2-\lambda)\left((2-\lambda)^2 - 1\right) = (2-\lambda)(4-4\lambda+\lambda^2-1) =$$

$$= (2-\lambda)(\lambda^2-4\lambda+3) = (2-\lambda)(\lambda-1)(\lambda-3) \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$$

$$\% = \begin{vmatrix} 3-\lambda & 0 & 3-\lambda \\ 0 & 1-\lambda & 1-\lambda \\ 1 & -1 & 2-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)\cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix} \overset{(-1)}{=} (3-\lambda)(1-\lambda) \overset{(R)}{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1-\lambda \end{vmatrix}} =$$

$$= (3-\lambda)(1-\lambda)\cdot 1\cdot(-1)^2 \begin{vmatrix} 1 & 1 \\ -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)(2-\lambda)$$

$\lambda_1 = 1$:

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ -x + x_2 - x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right)$$

$x_3 = s$

$2x_2 + 0\cdot s = 0 \Rightarrow x_2 = 0$

$x_1 + 0 + s = 0 \Rightarrow x_1 = -s$

$(-s, 0, s)^T,\ s \in \mathbb{R} - \{0\}$

$\lambda_2 = 2$

$$\begin{aligned} +x_2 + x_3 &= 0 \\ -x_1 \quad - x_3 &= 0 \\ x - x_2 \quad &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \overset{2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$x_3 = t$

$-x_2 - t = 0 \Rightarrow x_2 = -t$

$x_1 - (-t) = 0 \Rightarrow x_1 = -t$

$(-t, -t, t)^T,\ t \in \mathbb{R} - \{0\}$

$\lambda_3 = 3$: DÚ $\qquad (0, -u, u)^T,\ u \in \mathbb{R} - \{0\}$

b) $\begin{pmatrix} -1 & 4 & 3 \\ -2 & 5 & 3 \\ 2 & -4 & -2 \end{pmatrix}$

$$\begin{vmatrix} -1-\lambda & 4 & 3 \\ -2 & 5-\lambda & 3 \\ 2 & -4 & -2-\lambda \end{vmatrix} \% = (-1-\lambda)(5-\lambda)(-2-\lambda)+24+24-6(5-\lambda)+8(-2-\lambda)+12(-1-\lambda) =$$

$$= \dots = -\lambda^3+2\lambda^2-\lambda = -\lambda(\lambda^2-2\lambda+1) = -\lambda(\lambda-1)^2 = 0$$

$\lambda_1 = 0$

$\lambda_{2,3} = 1$

LR ↓

$$\% = \begin{vmatrix} 1-\lambda & 0 & 1-\lambda \\ 0 & 1-\lambda & 1-\lambda \\ 2 & -4 & -2-\lambda \end{vmatrix} = (1-\lambda)^2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & -4 & -2-\lambda \end{vmatrix} \begin{matrix} \\ \\ -2 \end{matrix} = (1-\lambda)^2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -4 & -4-\lambda \end{vmatrix} =$$

$$= (1-\lambda)^2 \left(1 \cdot (-1)^2 \begin{vmatrix} 1 & 1 \\ -4 & -4-\lambda \end{vmatrix} + 0 \cdots + 0 \cdots \right) = (1-\lambda)^2(-\lambda) = 0$$

$\lambda_1 = 0$:

$$\begin{aligned} -x_1 + 4x_2 + 3x_3 &= 0 \\ -2x_1 + 5x_2 + 3x_3 &= 0 \\ 2x_1 - 4x_2 - 2x_3 &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} -1 & 4 & 3 & 0 \\ -2 & 5 & 3 & 0 \\ 2 & -4 & -2 & 0 \end{array}\right) \begin{matrix} -2 & 2 \\ + & \\ & + \end{matrix} \sim \left(\begin{array}{ccc|c} -1 & 4 & 3 & 0 \\ 0 & -3 & -3 & 0 \\ \cancel{0} & \cancel{4} & \cancel{4} & \cancel{0} \end{array}\right)$$

$x_2 = s$

$-3s - 3x_3 = 0 \Rightarrow x_3 = -s$

$-x_1 + 4s + 3(-s) = 0 \Rightarrow x_1 = s$

$(s, s, -s)^T$, $s \in \mathbb{R} - \{0\}$

$\lambda_{2,3} = 1$

$$\begin{aligned} -2x_1 + 4x_2 + 3x_3 &= 0 \\ -2x_1 + 4x_2 + 3x_3 &= 0 \\ 2x_1 - 4x_2 - 3x_3 &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} 2 & -4 & -3 & 0 \\ \cancel{-2} & \cancel{4} & \cancel{3} & \cancel{0} \\ \cancel{-2} & \cancel{4} & \cancel{3} & \cancel{0} \end{array}\right)$$

$x_3 = t$

$x_2 = s$

$2x_1 - 4s - 3t = 0 \Rightarrow x_1 = 2s + \frac{3}{2}t$

$(2s + \frac{3}{2}t; s; t)^T$, $s, t \in \mathbb{R} - \{0\}$

$x_3 = 2t$

$x_2 = s$

$2x_1 - 4s - 3 \cdot 2t = 0 \Rightarrow x_1 = 2s + 3t$

$(2s + 3t, s, 2t)^T$; $s, t \in \mathbb{R} - \{0\}$

5. 1. Jsou dány vektory

a) $\vec{x} = 2\vec{u} + 3\vec{v}, \vec{y} = 3\vec{u} + m\vec{v}$,

b) $\vec{x} = m\vec{u} + \vec{v}, \vec{y} = 3\vec{u} + m\vec{v}$,

kde vektory $\vec{u}, \vec{v}$ jsou nekolineární. Určete $m \in \mathbb{R}$ tak (existuje-li), aby vektory $\vec{x}, \vec{y}$ byly kolineární.

b) $\vec{x} = m\vec{u} + \vec{v}$ $\quad$ $\vec{u}, \vec{v}$ – nekolineární!

$\vec{y} = 3\vec{u} + m\vec{v}$

i) $a\vec{x} = \vec{y}$

$a(m\vec{u} + \vec{v}) = 3\vec{u} + m\vec{v}$

$am\vec{u} + a\vec{v} = 3\vec{u} + m\vec{v}$

$am = 3 \wedge a = m$

$a^2 = 3 \Rightarrow a = \pm\sqrt{3}$

$m = \pm\sqrt{3}$

ii) $\alpha(\underbrace{m\vec{u} + \vec{v}}_{\vec{x}}) + \beta(\underbrace{3\vec{u} + m\vec{v}}_{\vec{y}}) = \vec{o}$

$\alpha m\vec{u} + \alpha\vec{v} + 3\beta\vec{u} + \beta m\vec{v} = \vec{o}$

$(\alpha m + 3\beta)\vec{u} + (\alpha + \beta m)\vec{v} = \vec{o}$ $\quad$ $\vec{u}, \vec{v}$ jsou nez.

$\Rightarrow \begin{aligned} \alpha m + 3\beta &= 0 \\ \alpha + m\beta &= 0 \end{aligned}$ $\quad$ $\begin{vmatrix} m & 3 \\ 1 & m \end{vmatrix} = 0$ $\quad$ $m^2 - 3 = 0$

$m^2 = 3 \Rightarrow m = \pm\sqrt{3}$

$\left(\begin{array}{cc|c} m & 3 & 0 \\ 1 & m & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & m & 0 \\ m & 3 & 0 \end{array}\right) \overset{-m}{\sim} \left(\begin{array}{cc|c} 1 & m & 0 \\ 0 & -m^2+3 & 0 \end{array}\right)$

5.

3. Jsou dány nekomplanární vektory $\vec{u}, \vec{v}, \vec{w}$. Rozhodněte, zda následující vektory $\vec{a}, \vec{b}, \vec{c}$ jsou lineárně závislé či nezávislé, je-li

a) $\vec{a} = \vec{u} + \vec{v}, \vec{b} = \vec{u} + \vec{w}, \vec{c} = \vec{v} + \vec{w}$,

b) $\vec{a} = \vec{u} + \vec{v}, \vec{b} = \vec{v} - \vec{w}, \vec{c} = \vec{u} + \vec{w}$,

$$\alpha \vec{a} + \beta \vec{b} + \gamma \vec{c} = \vec{o}$$

$$\alpha(\vec{u} + \vec{v}) + \beta(\vec{u} + \vec{w}) + \gamma(\vec{v} + \vec{w}) = \vec{o}$$

$$\alpha \vec{u} + \alpha \vec{v} + \beta \vec{u} + \beta \vec{w} + \gamma \vec{v} + \gamma \vec{w} = \vec{o}$$

$$(\alpha + \beta)\vec{u} + (\alpha + \gamma)\vec{v} + (\beta + \gamma)\vec{w} = \vec{o}$$

$\vec{u}, \vec{v}, \vec{w}$ - nekomplanární $\Rightarrow$

$$\alpha + \beta = 0$$
$$\alpha + \gamma = 0$$
$$\beta + \gamma = 0 \Rightarrow \gamma = -\beta$$

$$\left.\begin{aligned} \alpha + \beta &= 0 \\ \alpha - \beta &= 0 \end{aligned}\right) +$$

$\Rightarrow \alpha = 0$

$\beta = 0$

$\gamma = 0$

$\Rightarrow$ LNZ

5. 4. Jsou dány vektory

(a) $\vec{x} = \vec{u} + \vec{v} - 3\vec{w}, \vec{y} = 3\vec{u} + \vec{v} + \vec{w}, \vec{z} = \vec{u} + m\vec{v} + 2\vec{w},$

b) $\vec{x} = \vec{u} - \vec{v} - \vec{w}, \vec{y} = \vec{u} - \vec{v} + \vec{w}, \vec{z} = m\vec{u} + 2\vec{v},$

kde $\vec{u}, \vec{v}, \vec{w}$ jsou nekomplanární vektory. Určete číslo $m \in \mathbb{R}$ tak, aby vektory $\vec{x}, \vec{y}, \vec{z}$ byly komplanární.

a) $\vec{x} = \vec{u} + \vec{v} - 3\vec{w}$

$\vec{y} = 3\vec{u} + \vec{v} + \vec{w}$

$\vec{z} = \vec{u} + m\vec{v} + 2\vec{w}$

$a\vec{x} + b\vec{y} = \vec{z}$

$a(\vec{u} + \vec{v} - 3\vec{w}) + b(3\vec{u} + \vec{v} + \vec{w}) = \vec{u} + m\vec{v} + 2\vec{w}$

$\vec{u}: \quad a + 3b = 1$

$\vec{v}: \quad a + b = m$

$\vec{w}: \quad -3a + b = 2$

$\left.\begin{array}{l} a + 3b = 1 \\ -3a + b = 2 \end{array}\right\} \cdot 3 \Rightarrow 10b = 5 \Rightarrow b = \frac{1}{2}$

$a + 3 \cdot \frac{1}{2} = 1 \Rightarrow a = -\frac{1}{2}$

$-\frac{1}{2} + \frac{1}{2} = m \Rightarrow m = 0$

2. Jsou dány vektory

   a) $\vec{x} = -2\vec{u} + 3\vec{v} + m\vec{w}, \vec{y} = n\vec{u} - 6\vec{v} + 2\vec{w}$,

   b) $\vec{x} = 3\vec{u} - \vec{v} + \vec{w}, \vec{y} = \vec{u} + m\vec{v} - n\vec{w}$,

   kde $\vec{u}, \vec{v}, \vec{w}$ jsou nekomplanární vektory. Určete čísla $m, n \in \mathbb{R}$ tak, aby vektory $\vec{x}, \vec{y}$ byly kolineární.