4. Užitím smíšeného součinu rozhodněte, zda vektory $\vec{u}, \vec{v}, \vec{w}$ jsou komplanární, jestliže $\vec{u} = \vec{a} - 2\vec{b} + \vec{c}, \vec{v} = 3\vec{a} + \vec{b} - 2\vec{c}, \vec{w} = 7\vec{a} + 14\vec{b} - 13\vec{c}$, kde $\vec{a}, \vec{b}, \vec{c}$ jsou nekomplanární vektory.

$[\vec{u}, \vec{v}, \vec{w}] = (\vec{u} \times \vec{v}) \cdot \vec{w}$

$[\vec{v}, \vec{w}, \vec{u}] = \vec{u} \cdot (\vec{v} \times \vec{w})$

$[\vec{u}, \vec{v}, \vec{w}] = \nearrow$

$\vec{u} = \vec{a} - 2\vec{b} + \vec{c}$

$\vec{v} = 3\vec{a} + \vec{b} - 2\vec{c}$

$\vec{w} = 7\vec{a} + 14\vec{b} - 13\vec{c}$

$\vec{a}, \vec{b}, \vec{c} \ldots$ nekomplanární

$$[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{a} - 2\vec{b} + \vec{c}) \cdot [(3\vec{a} + \vec{b} - 2\vec{c}) \times (7\vec{a} + 14\vec{b} - 13\vec{c})] =$$

$$= (\vec{a} - 2\vec{b} + \vec{c}) \cdot (\cancel{21\, \vec{a} \times \vec{a}} + 42\, \vec{a} \times \vec{b} - 39\, \vec{a} \times \vec{c} + \underbrace{7\, \vec{b} \times \vec{a}}_{-7\, \vec{a} \times \vec{b}} + \cancel{14\, \vec{b} \times \vec{b}} - 13\, \vec{b} \times \vec{c} -$$

$$- \underbrace{14\, \vec{c} \times \vec{a}}_{+14\, \vec{a} \times \vec{c}} - \underbrace{28\, \vec{c} \times \vec{b}}_{28\, \vec{b} \times \vec{c}} + \cancel{26\, \vec{c} \times \vec{c}}) =$$

$$= (\vec{a} - 2\vec{b} + \vec{c})(35\, \vec{a} \times \vec{b} - 25\, \vec{a} \times \vec{c} + 15\, \vec{b} \times \vec{c}) =$$

$$= 35\, \cancel{\vec{a} \cdot (\vec{a} \times \vec{b})} - 25\, \cancel{\vec{a} (\vec{a} \times \vec{c})} + 15\, \vec{a} \cdot (\vec{b} \times \vec{c}) - 70\, \cancel{\vec{b} \cdot (\vec{a} \times \vec{b})} + 50\, \vec{b} \cdot (\vec{a} \times \vec{c}) - 30\, \cancel{\vec{b} \cdot (\vec{b} \times \vec{c})} +$$

$$+ 35\, \vec{c} \cdot (\vec{a} \times \vec{b}) - 25\, \cancel{\vec{c} (\vec{a} \times \vec{c})} + 15\, \cancel{\vec{c} \cdot (\vec{b} \times \vec{c})} = 15 [\vec{a}, \vec{b}, \vec{c}] + 50 [\vec{b}, \vec{a}, \vec{c}] + 35 [\vec{c}, \vec{a}, \vec{b}] =$$

$$= 15 [\vec{a}, \vec{b}, \vec{c}] - 50 [\vec{a}, \vec{b}, \vec{c}] + 35 [\vec{a}, \vec{b}, \vec{c}] = 0 \implies \vec{u}, \vec{v}, \vec{w} \text{ JSOU KOMPLANÁRNÍ}$$

7.4 Dvojný vektorový součin geometrických vektorů ve $\mathbb{V}(\mathbb{E}_3)$

2. Užitím vztahu (I) dokažte, že platí:

   a) $(\vec{b} \times \vec{c}) \cdot [(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})] = [\vec{a}, \vec{b}, \vec{c}]^2$,

   b) $\vec{a} \times [\vec{b} \times (\vec{c} \times \vec{d})] = [\vec{a} \cdot (\vec{c} \times \vec{d})] \cdot \vec{b} - (\vec{a} \cdot \vec{b}) \cdot (\vec{c} \times \vec{d})$,

   c) $\vec{a} \times [\vec{b} \times (\vec{c} \times \vec{d})] = (\vec{b} \cdot \vec{d}) \cdot (\vec{a} \times \vec{c}) - (\vec{b} \cdot \vec{c}) \cdot (\vec{a} \times \vec{d})$,

   d) $(\vec{a} \times \vec{b}) \cdot [(\vec{c} \times \vec{d}) \times (\vec{e} \times \vec{f})] =$
   $= [(\vec{a} \times \vec{b}) \cdot \vec{e}] \cdot [\vec{f} \cdot (\vec{c} \times \vec{d})] - [(\vec{a} \times \vec{b}) \cdot \vec{f}] \cdot [\vec{e} \cdot (\vec{c} \times \vec{d})]$.

$$(I)\quad \vec a \times (\vec b \times \vec c) = \overset{\text{VEKTOR}}{\vec b} \cdot (\underbrace{\vec a \cdot \vec c}_{\in R}) - \overset{\text{VEKTOR}}{\vec c} \cdot (\underbrace{\vec a \cdot \vec b}_{\in R})$$

a)

$$(\vec c \times \vec a) \times (\vec a \times \vec b) \overset{(I)}{=} \vec a \cdot ((\vec c \times \vec a) \cdot \vec b) - \vec b \cdot ((\vec c \times \vec a) \cdot \vec a) = \vec a \cdot [\vec c, \vec a, \vec b] - \vec b \underbrace{[\vec c, \vec a, \vec a]}_{=0} =$$

$$= \vec a \cdot [\vec a, \vec b, \vec c]$$

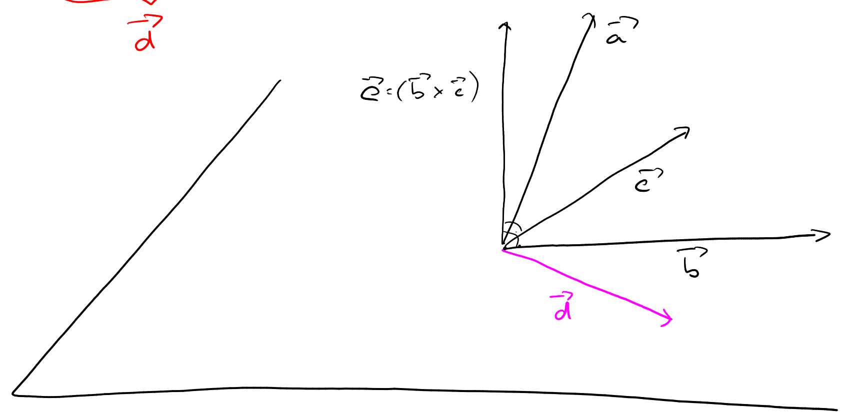
$$(\vec b \times \vec c) \cdot \vec a \cdot [\vec a, \vec b, \vec c] = [\vec b, \vec c, \vec a] \cdot [\vec a, \vec b, \vec c] = [\vec a, \vec b, \vec c]^2$$

$$\underbrace{\vec a \times (\underbrace{\vec b \times \vec c}_{\vec e})}_{\vec d}$$

$\vec e \perp \vec b$

$\vec e \perp \vec c$

$\vec d$ - LEŽÍ V ROVINĚ $(\vec b, \vec c)$



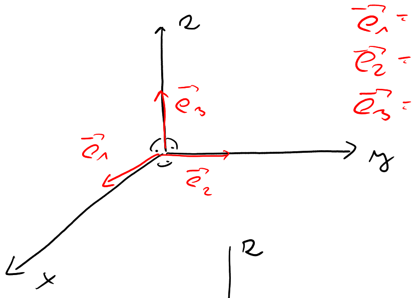
7.5 Operace s geometrickými vektory v uspořádané pozitivní ortonormální bázi $E = (\vec e_1, \vec e_2, \vec e_3)$ ve $\mathbb{V}(\mathbb{E}_3)$

1. Určete směrové kosiny vektorů
   a) $\vec a = \vec e_2 + 3\vec e_3$,
   b) $\vec b = 2\vec e_1 - 3\vec e_2 - \vec e_3$.

$\vec e_1 = \vec i = (1,0,0)$

$\vec e_2 = \vec j = (0,1,0)$

$\vec e_3 = \vec k = (0,0,1)$



$(\cos\alpha, \cos\beta, \cos\gamma) = \vec a_0$ ... jednotkový vektor

$\vec a$

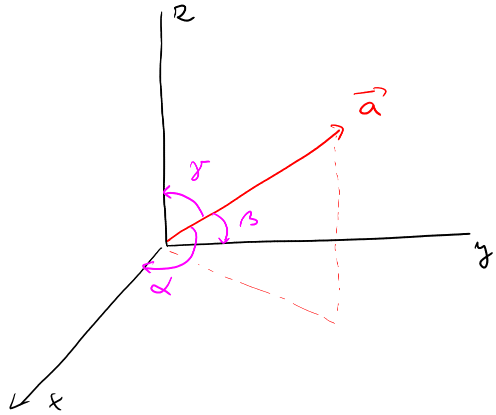
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 = \left[\frac{a_1^2 + a_2^2 + a_3^2}{\|\vec a\|^2}\right]$$

$$\cos\alpha = \frac{\vec a \cdot \vec e_1}{\|\vec a\|\|\vec e_1\|} = \frac{a_1}{\|\vec a\|}$$

$$\cos\beta = \frac{a_2}{\|\vec a\|}$$

$$\cos\gamma = \frac{a_3}{\|\vec a\|}$$

$$\vec a = a_1\cdot\vec e_1 + a_2\vec e_2 + a_3\cdot\vec e_3 = (a_1, a_2, a_3)$$



a) $\vec a = \vec e_2 + 3\vec e_3$

$\vec a = (0, 1, 3)$

$$\cos\alpha = \frac{\vec a \cdot \vec e_1}{\|\vec a\|\|\vec e_1\|} = \frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}} = \frac{0}{\sqrt{0+1+9}} = \frac{0}{\sqrt{10}} = 0$$

$$\cos\beta = \frac{a_2}{\|\vec a\|} = \frac{1}{\sqrt{10}}$$

$$\cos\gamma = \frac{a_3}{\|\vec a\|} = \frac{3}{\sqrt{10}}$$

b) $\vec b = 2\vec e_1 - 3\vec e_2 - \vec e_3 = (2, -3, -1)$ $\quad \|\vec b\| = \sqrt{2^2 + (-3)^2 + (-1)^2} = \sqrt{14}$

$$\cos\alpha = \frac{2}{\sqrt{14}}$$

$$\cos\beta = -\frac{3}{\sqrt{14}}$$

$$\cos\gamma = -\frac{1}{\sqrt{14}}$$

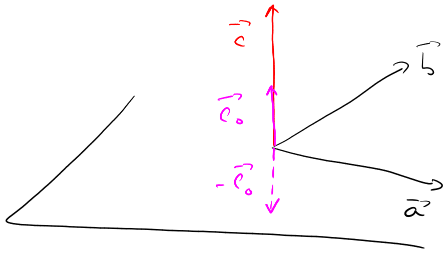
7.5. 2. Najděte jednotkový vektor $\vec{c}^0$ kolmý k vektorům $\vec{a} = 2\vec{e}_1 - \vec{e}_2 + \vec{e}_3, \vec{b} = \vec{e}_1 + 2\vec{e}_2 - \vec{e}_3$.

$\vec{a} = (2,-1,1)\ ,\ \vec{b} = (1,2,-1)$

I. (lepší) $\quad \vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \left(\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}, -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \right) = (-1,3,5)$

$$\vec{c}_0 = \frac{\vec{c}}{\|\vec{c}\|} = \underline{\underline{\pm \frac{1}{\sqrt{35}}(-1,3,5)}}$$

$$\|\vec{c}\| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{35}$$



$$\vec{c}_{01} = \left(-\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}\right)$$

$$\vec{c}_{02} = \left(\frac{1}{\sqrt{35}}, -\frac{3}{\sqrt{35}}, -\frac{5}{\sqrt{35}}\right)$$

II. $\quad \left.\begin{array}{l} \vec{a} \cdot \vec{c} = 0 \\ \vec{b} \cdot \vec{c} = 0 \end{array}\right\}$ SYSTÉM 2 ROVNIC O TŘECH NEZNÁMÝCH $\Rightarrow \infty$ ŘEŠENÍ

1 VOLNÝ PARAMETR

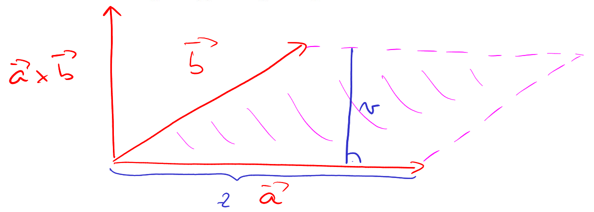
$\vec{c} = (c_1, c_2, c_3)$

$$\left.\begin{array}{l} 2c_1 - c_2 + c_3 = 0 \\ c_1 + 2c_2 - c_3 = 0 \end{array} \middle|\ c_3 = 1 \ \right| \Bigg\} \Rightarrow \begin{array}{l} 2c_1 - c_2 = -1 \quad /\cdot 2 \\ c_1 + 2c_2 = 1 \end{array} \qquad 5c_1 = -1 \Rightarrow c_1 = -\frac{1}{5}, \quad c_2 = \frac{3}{5}$$

$$\Rightarrow \vec{c} = \left(-\frac{1}{5}, \frac{3}{5}, 1\right) \longleftrightarrow (1,-3,-5) \longrightarrow (-1,3,5)$$

$$\Rightarrow \vec{c}_0 = \pm \frac{1}{\sqrt{35}}(1,-3,-5)$$

7.5. 3. Vypočtěte plošný obsah a výšku na stranu určenou vektorem $\vec{a}$ rovnoběžníka setrojeného nad vektory $\vec{a} = 2\vec{e}_2 + \vec{e}_3, \vec{b} = \vec{e}_1 + 2\vec{e}_3$.



$\vec{a} = (0, 2, 1)$

$\vec{b} = (1, 0, 2)$

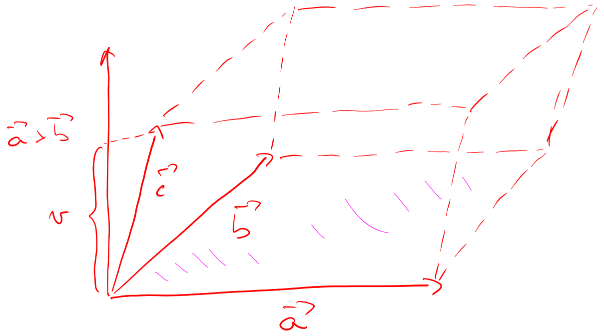
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \left(\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix}, - \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} \right) = (4, 1, -2)$$

$$P = \|\vec{a} \times \vec{b}\| = \|(0,2,1) \times (1,0,2)\| = \|(4,1,-2)\| = \sqrt{4^2 + 1^2 + (-2)^2} = \sqrt{21}$$

$$P = z \cdot v \qquad z = \|\vec{a}\| = \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5}$$

$$v = \frac{P}{z} = \frac{\sqrt{21}}{\sqrt{5}} = \sqrt{\frac{21}{5}}$$

7.5. 4. Vypočtěte objem rovnoběžnostěnu sestrojeného nad vektory $\vec{a}, \vec{b}, \vec{c}$, plošný obsah stěny sestrojené nad vektory $\vec{a}, \vec{b}$ a velikost výšky na tuto stěnu, je-li $\vec{a} = 3\vec{e}_1 + 2\vec{e}_2, \vec{b} = 2\vec{e}_1 + 3\vec{e}_2, \vec{c} = \vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$.



$$v = \|\vec{c}_{\vec{a}\times\vec{b}}\| = \|\vec{c}\| \cdot \cos(\vec{c}, \vec{a}\times\vec{b}) = \frac{\vec{c}\cdot(\vec{a}\times\vec{b})}{\|\vec{a}\times\vec{b}\|}$$

$V = |[\vec{a}, \vec{b}, \vec{c}]|$, $P = \|\vec{a}\times\vec{b}\|$

$V = P \cdot v$

$v = \frac{V}{P}$

$\vec{a} = (3, 2, 0)$
$\vec{b} = (2, 3, 0)$
$\vec{c} = (1\ 2\ 3)$

a) $$V = |[\vec{a}, \vec{b}, \vec{c}]| = \begin{array}{c} \quad\quad\quad\downarrow LR \\ \begin{vmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{vmatrix} \end{array} = 3 \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 15$$

b) $$P = \|\vec{a}\times\vec{b}\| = \|(0, 0, 5)\| = \sqrt{25} = 5$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix} = \left(\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix}, -\begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} \right) = (0, 0, 5)$$

$$v = \frac{V}{P} = \frac{15}{5} = 3$$

7.5. 5. Určete úhel vektorů $\vec{a} = \vec{e}_1 + \vec{e}_2 - 4\vec{e}_3, \vec{b} = \vec{e}_1 - 2\vec{e}_2 + 2\vec{e}_3$.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\varphi \Rightarrow \cos\varphi = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}$$

$\vec{a} = (1, 1, -4)$

$\vec{b} = (1, -2, 2)$

$$\vec{a} \cdot \vec{b} = (1,1,-4) \cdot (1,-2,2) = 1 \cdot 1 + 1 \cdot (-2) + (-4) \cdot 2 = -9$$

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{1^2 + 1^2 + (-4)^2} = \sqrt{18} = \sqrt{2} \cdot 3$$

$$\|\vec{b}\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{-9}{\sqrt{2} \cdot 3 \cdot 3} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow \varphi = \frac{3}{4}\pi = 135^\circ$$

6. Vypočtěte $\vec{a} \times (\vec{b} \times \vec{c})$, kde $\vec{a} = 2\vec{e}_1, \vec{b} = 3\vec{e}_2, \vec{c} = \vec{e}_1 + \vec{e}_3$

a) přímým výpočtem,

b) užitím vztahu $(I) : \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{a} \cdot \vec{c}) - \vec{c} \cdot (\vec{a} \cdot \vec{b})$,

c) užitím determinantů.

$\vec{a} = (2,0,0)$
$\vec{b} = (0,3,0)$
$\vec{c} = (1,0,1)$

POZN: $A \cdot (B \cdot C$ (matice)) ✓ SPRÁVNÝ ZÁPIS

$\vec{a} \cdot \vec{b} \cdot \vec{c}$ ✗ NESPRÁVNÝ ZÁPIS

$\vec{a} \times \vec{b} \times \vec{c}$ ✗ NESPRÁVNÝ ZÁPIS

} NUTNÉ ZÁVORKOVAT

a) $$\vec{a} \times (\vec{b} \times \vec{c}) = 2\vec{e}_1 \times (3\vec{e}_2 \times (\vec{e}_1 + \vec{e}_3)) = 2\vec{e}_1 (3 \underbrace{\vec{e}_2 \times \vec{e}_1}_{-e_3} + 3 \underbrace{\vec{e}_2 \times \vec{e}_3}_{\vec{e}_1}) =$$

$$= -6 \underbrace{\vec{e}_1 \times e_3}_{-\vec{e}_2} + 6 \underbrace{\vec{e}_1 \times \vec{e}_1}_{0} = \underline{\underline{6\vec{e}_2}}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (2,0,0) \times (\underbrace{(0,3,0) \times (1,0,1)}_{(3,0,-3)}) = (2,0,0) \times (3,0,-3) = (0,6,0) =$$

$$= \underline{\underline{6\vec{e}_2}}$$

b) $$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{a} \cdot \vec{c}) - \vec{c} \cdot (\vec{a} \cdot \vec{b}) = 3\vec{e}_2 \cdot (\underbrace{2 \cdot \vec{e}_1 \cdot \vec{e}_1}_{2 \cdot \|\vec{e}_1\| = 1} + \underbrace{2\vec{e}_1 \cdot e_3}_{=0}) - \underbrace{(\vec{e}_1 + \vec{e}_3) \cdot (\underbrace{2\vec{e}_1 \cdot 3\vec{e}_2}_{=0})}_{=0} =$$

$$= \underline{\underline{6\vec{e}_2}}$$

c) $$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (3,0,-3) = 3\vec{e}_1 - 3\vec{e}_3$$

$$a \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 2 & 0 & 0 \\ 3 & 0 & -3 \end{vmatrix} = (0,6,0) = \underline{\underline{6\vec{e}_2}}$$