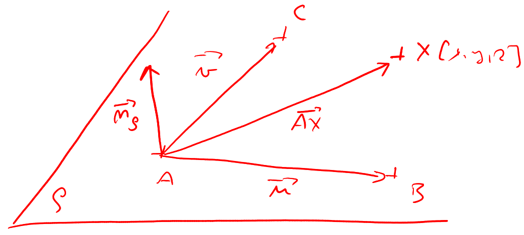
7.6 Aplikace vektorové algebry v analytické geometrii v prostoru $\mathbb{E}_3$ (v kartézské souřadnicové soustavě)

1. Určete
   a) parametrický,
   b) obecný,
   c) úsekový

   tvar rovnice roviny ρ, jestliže $\rho = [A, \overrightarrow{AB}, \overrightarrow{AC}]$, kde $A = [2,3,1]$, $B = [3,1,4]$, $C = [2,1,5]$.



$\rho = [A, \vec{u}, \vec{v}]$, $\vec{u} = \overrightarrow{AB}$, $\vec{v} = \overrightarrow{AC}$

a) PARAMETRICKÝ TVAR

$X = A + k\vec{u} + l\vec{v}$

$\vec{u} = \overrightarrow{AB} = B - A = (1,-2,3)$, $\vec{v} = \overrightarrow{AC} = C - A = (0,-2,4)$

$$\rho: \quad \begin{aligned} x &= 2 + k \\ y &= 3 - 2k - 2l \\ z &= 1 + 3k + 4l \end{aligned}$$

b) OBECNÝ TVAR: $ax + by + cz + d = 0$; $\vec{n}_\rho = (a,b,c)$

i) $\vec{n}_\rho \sim \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 0 & -2 & 4 \end{vmatrix} = \left(\begin{vmatrix} -2 & 3 \\ -2 & 4 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ 0 & -2 \end{vmatrix} \right) = (-2,-4,-2) \sim (1,2,1)$

$\rho: x + 2y + z + d = 0$

$A \in \rho: 2 + 2\cdot 3 + 1 + d = 0$

$d = -9$

$\rho: x + 2y + z - 9 = 0$

ii) $[\overrightarrow{AX}, \vec{u}, \vec{v}] = 0$, $\overrightarrow{AX} = (x-2, y-3, z-1)$

$\overrightarrow{LR}$ $\begin{vmatrix} x-2 & y-3 & z-1 \\ 1 & -2 & 3 \\ 0 & -2 & 4 \end{vmatrix} = 0 \Rightarrow (x-2)\cdot(-2) - (y-3)\cdot 4 + (z-1)\cdot(-2) = 0$

$-2x + 4 - 4y + 12 - 2z + 2 = 0$

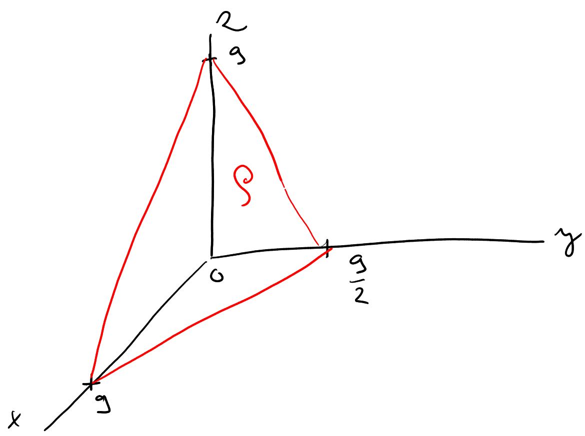
$-2x - 4y - 2z + 18 = 0 \quad /:-2$

$x + 2y + z - 9 = 0$

c) $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

$\rho: x + 2y + z = 9 \quad /:9$

$\dfrac{x}{9} + \dfrac{y}{\frac{9}{2}} + \dfrac{z}{9} = 1$



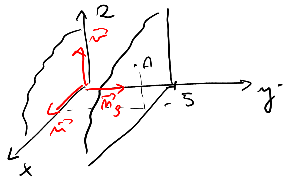
2. Určete rovnici roviny, která

a) je rovnoběžná s rovinou (x, z) a prochází bodem $A = [2, -5, 3]$,

b) prochází osou z a bodem $A = [-3, 1, -2]$,

c) je rovnoběžná s osou x a prochází body $B = [4, 0, -2]$, $C = [5, 1, 7]$.

a)



i) $\vec{u} = (1,0,0)$ $\quad \vec{AX} = (x-2,\ y+5,\ z-3)$

$\vec{v} = (0,0,1)$

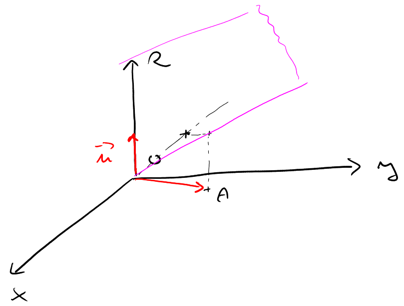
$$\begin{vmatrix} x-2 & y+5 & z-3 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \Rightarrow \cancel{(x-2)}\cdot 0 - (y+5)\cdot 1 + \cancel{(z-3)}\cdot 0 = 0$$

$$\Rightarrow \underline{\underline{y+5=0}}$$

ii) $\vec{n}_\rho = (0,1,0) \Rightarrow \rho:\ y + d = 0$

$A \in \rho:\ -5 + d = 0 \Rightarrow d = 5$ $\Big\}$ $\underline{\underline{y+5=0}}$

b)



$\vec{u} = (0,0,1)$

$\vec{v} = \vec{OA} = (-3, 1, -2)$

$$\vec{n}_\rho \sim \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -3 & 1 & -2 \end{vmatrix} = \left(\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ -3 & -2 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ -3 & 1 \end{vmatrix} \right) = (-1, -3, 0)$$

$\rho:\ -x - 3y + d = 0$

$A \in \rho:\ +3 - 3 + d = 0 \Rightarrow d = 0$ $\Big\}$ $-x - 3y = 0$

$\underline{\underline{x + 3y = 0}}$

c) $\vec{u} = (1,0,0)$, $\vec{v} = \vec{BC} = (1, 1, 9)$

$$\vec{n}_\rho \sim \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 1 & 1 & 9 \end{vmatrix} = \left(\begin{vmatrix} 0 & 0 \\ 1 & 9 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 1 & 9 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \right) = (0, -9, 1)$$

$\rho:\ -9y + z + d = 0$

$B \in \rho:\ 0 - 2 + d = 0 \Rightarrow d = 2$ $\Big\}$ $\rho:\ -9y + z + 2 = 0$

$\underline{\underline{9y - z - 2 = 0}}$

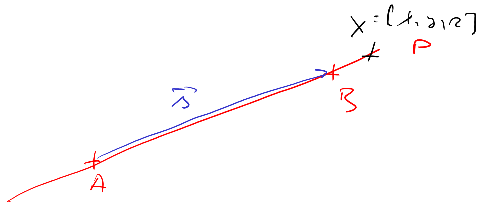
3. Určete

a) parametrický,
b) kanonický

tvar rovnice přímky p,

c) přímku p jako průsečnici různoběžných rovin,

je-li $p = [A, \overrightarrow{AB}]$, kde $A = [2, 9, 3]$, $B = [5, 3, 11]$.



a) i) $\vec{s} = \overrightarrow{AB} = (3, -6, 8)$ $\quad X = A + t\vec{s}$

$$\begin{aligned} x &= 2 + 3t \\ y &= 9 - 6t \quad , t \in \mathbb{R} \\ z &= 3 + 8t \end{aligned}$$

ii) $X \in p$; $\overrightarrow{AX} = t \cdot \vec{s} \Rightarrow (x-2, y-9, z-3) = t(3, -6, 8)$

$$\left.\begin{aligned} t = \frac{x-2}{3} &\Leftarrow x - 2 = 3t \\ t = \frac{y-9}{-6} &\Leftarrow y - 9 = -6t \\ t = \frac{z-3}{8} &\Leftarrow z - 3 = 8t \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 2 + 3t \\ y &= 9 - 6t \quad ; t \in \mathbb{R} \\ z &= 3 + 8t \end{aligned}$$

b) $\dfrac{x - x_0}{s_1} = \dfrac{y - y_0}{s_2} = \dfrac{z - z_0}{s_3} = (t)$ $\quad [x_0, y_0, z_0] = A$, $(s_1, s_2, s_3) = \vec{s}$

$$p: \frac{x-2}{3} = \frac{y-9}{-6} = \frac{z-3}{8}$$

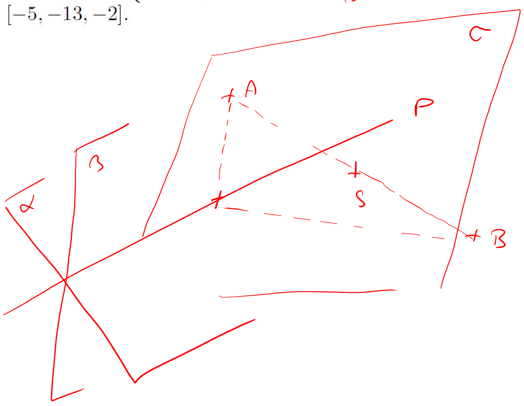
c) $p: \begin{cases} \dfrac{x-2}{3} = \dfrac{y-9}{-6} \\ \dfrac{y-9}{-6} = \dfrac{z-3}{8} \end{cases}$

$-6(x-2) = 3(y-9) \Rightarrow -6x + 12 - 3y + 27 = 0$
$-6x - 3y + 39 = 0 \quad /:(-3)$
$2x + y - 13 = 0$

$8(y-9) = -6(z-3) \Rightarrow 8y - 72 + 6z - 18 = 0$
$8y + 6z - 90 = 0 \quad /:2$
$4y + 3z - 45 = 0$

$p: \begin{cases} 2x + y - 13 = 0 \\ 4y + 3z - 45 = 0 \end{cases}$

4. Na přímce $p : \begin{cases} x + 2y + z - 1 = 0 & \alpha \\ 3x - y + 4z - 29 = 0 & \beta \end{cases}$ určete bod, který má stejnou vzdálenost od bodů $A = [3, 11, 4]$, $B = [-5, -13, -2]$.



σ : ROVINA SYMETRIE ÚSEČKY AB

$\vec{n}_\sigma \sim \vec{AB} = (-8, -24, -6) \sim (4, 12, 3)$

$$S = \frac{A+B}{2} = [-1, -1, 1]$$

$\sigma : \; 4x + 12y + 3z + d = 0$

$S \in \sigma : \; -4 - 12 + 3 + d = 0 \Rightarrow d = 13$

$\sigma : \; 4x + 12y + 3z + 13 = 0$

$X \in \sigma \wedge X \in p$:
$$\begin{aligned} x + 2y + z - 1 &= 0 \\ 3x - y + 4z - 29 &= 0 \\ 4x + 12y + 3z + 13 &= 0 \end{aligned}$$

i) $\Rightarrow x = 1 - 2y - z$

$$\begin{aligned} -7y + z &= 26 \\ 4y - z &= -17 \end{aligned} \Big\} + \quad -3y = 9, \quad y = -3$$

$z = 5$

$x = 2$

$X = [2, -3, 5]$

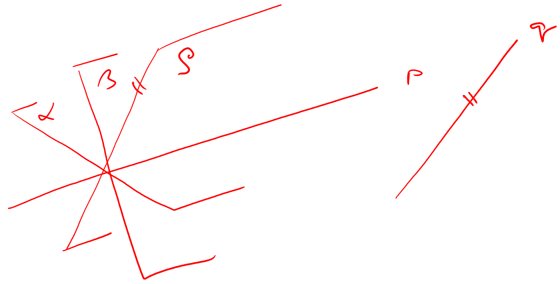
ii) $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 3 & -1 & 4 & 0 \\ 4 & 12 & 3 & 0 \end{array}\right) \sim$ GEM $\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -7 & 1 & 26 \\ 0 & 0 & -3 & -15 \end{array}\right) \Rightarrow \begin{aligned} z &= 5 \\ y &= -3 \\ x &= 2 \end{aligned} \qquad X = [2, -3, 5]$

DÚ

5. Určete rovnici roviny ρ, která prochází přímkou $p: \begin{cases} 2x - 3y + 2z - 6 = 0 & \alpha \\ 5x + y - 10z + 1 = 0 & \beta \end{cases}$ a je rovnoběžná s přímkou

$$q: \frac{x-3}{4} = \frac{y}{5} = \frac{z+2}{3}.$$

Úlohu řešte a) svazkem rovin, b) bez užití svazku rovin.



a) $\rho: k(2x-3y+2z-6) + l(5x+y-10z+1) = 0$

$(2k+5l)x + (-3k+l)y + (2k-10l)z + (-6k+l) = 0$

$\rho \parallel q \Rightarrow \vec{n}_\rho \perp \vec{s}_q \; ; \; \vec{s}_q = (4,5,3)$

$\vec{n}_\rho \cdot \vec{s}_q = 0 \Rightarrow (2k+5l, -3k+l, 2k-10l) \cdot (4,5,3) = 0$

$8k + 20l - 15k + 5l + 6k - 30l = 0$

$-k - 5l = 0$ např. $k = 5 \Rightarrow l = -1$

$\rho: 5x - 16y + 20z - 31 = 0$

b) $q: A = [3,0,-2] \; ; \; \vec{s}_q = (4,5,3)$

$\vec{s}_p = \vec{n}_\alpha \times \vec{n}_\beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 2 \\ 5 & 1 & -10 \end{vmatrix} = \dots$ DÚ $= (28, 30, 17)$

$\rho: X = A + k\vec{s}_p + l\vec{s}_q$

$x = 3 + 28k + 4l$
$y = 0 + 30k + 5l, \quad k, l \in \mathbb{R}$
$z = -2 + 17k + 3l$

$\vec{n}_\rho = \vec{s}_p \times \vec{s}_q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 28 & 30 & 17 \\ 4 & 5 & 3 \end{vmatrix} = \left(\begin{vmatrix} 30 & 17 \\ 5 & 3 \end{vmatrix}, -\begin{vmatrix} 28 & 17 \\ 4 & 3 \end{vmatrix}, \begin{vmatrix} 28 & 30 \\ 4 & 5 \end{vmatrix} \right) = (5, -16, 20)$

$\rho: 5x - 16y + 20z + d = 0$

$P \subset \rho \; ; \; P \in p$ např. $z = 0 \Rightarrow \begin{cases} 2x - 3y = 6 \\ 5x + y = -1 \; / \cdot 3 \end{cases} \quad 17x = 3 \Rightarrow x = \frac{3}{17}$

$y = -1 - \frac{15}{17} = -\frac{32}{17}$

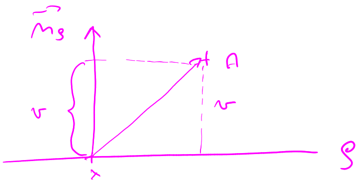
$P = \left[\frac{3}{17}, -\frac{32}{17}, 0\right]$

$P \in \rho: \frac{3}{17} \cdot 5 + \frac{32}{17} \cdot 16 + 20 \cdot 0 + d = 0 \Rightarrow d = \frac{-15 - 512}{17} = -\frac{527}{17} = -31$

6. Vypočtěte vzdálenost $v(A,\rho)$ bodu $A = [4,2,2]$ od roviny ρ a vzdálenost $v(A,q)$ bodu A od přímky q, kde ρ je rovina a q je přímka z příkladu 5.

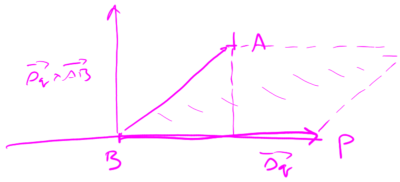
$\rho: 5x - 16y + 20z - 31 = 0$

$$q: \begin{aligned} x &= 3 + 4t \\ y &= 0 + 5t, \quad t \in \mathbb{R} \\ z &= -2 + 3t \end{aligned}$$



$$v(A,\rho) = \frac{|a x_A + b y_A + c z_A + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$v(A,q) = \frac{\| \vec{s}_q \times \vec{AB} \|}{\| \vec{s}_q \|}$$



a) $$v(A,\rho) = \frac{|5\cdot 4 - 16\cdot 2 + 20\cdot 2 - 31|}{\sqrt{5^2 + (-16)^2 + 20^2}} = \frac{|-3|}{\sqrt{681}} = \underline{\underline{\frac{3}{\sqrt{681}}}} \doteq 0{,}115$$

b) $\vec{AB} = (-1,-2,-4)$; $$\vec{s}_q \times \vec{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 5 & 3 \\ -1 & -2 & -4 \end{vmatrix} = \underset{\text{DU}}{\cdots} = (-14, 13, -3)$$

$$v(A,q) = \frac{\|(-14,13,-3)\|}{\|(4,5,3)\|} = \frac{\sqrt{(-14)^2 + 13^2 + (-3)^2}}{\sqrt{4^2 + 5^2 + 3^2}} = \frac{\sqrt{374}}{\sqrt{50}} = \underline{\underline{\sqrt{\frac{374}{50}}}} \doteq 2{,}735$$