Příklad 9.2. Rozložte v součet polynomů s parciálními zlomky:

6. $f : y = \dfrac{15x^2 - 2x + 6}{6x^3 + 12x}$

$$\left.\begin{array}{l} \text{st } g_1 = 2 \\ \text{st } g_2 = 3 \end{array}\right\} \Rightarrow \text{ryzí}$$

$$f(x) = \frac{15x^2 - 2x + 6}{6x(x^2+2)} = \frac{A}{x} + \frac{Bx + C}{x^2+2}$$

$$\begin{aligned} 15x^2 - 2x + 6 &= A \cdot 6(x^2+2) + (Bx + C)6x \\ &= 6Ax^2 + 12A + 6Bx^2 + 6Cx \end{aligned}$$

I. metoda dosazovací

$$\begin{array}{lll} x = 0: & 6 = 12A & \Rightarrow A = 1/2 \\ x = 1: & 19 = 18A + 6B + 6C & 6B + 6C = 10 \\ x = -1: & 23 = 18A + 6B - 6C & 6B - 6C = 14 \end{array} \quad \Big\} + \quad \begin{array}{l} 12B = 24 \\ B = 2 \\ C = -1/3 \end{array}$$

$$f(x) = \underline{\underline{\frac{1}{2x} + \frac{2x - \frac{1}{3}}{x^2+2}}} = \underline{\underline{\frac{1}{2x} + \frac{6x-1}{3(x^2+2)}}}$$

II. metoda neurčitých koeficientů

$$\begin{array}{lll} x^2: & 15 = 6A + 6B & \Rightarrow B = 2 \\ x^1: & -2 = 6C & \Rightarrow C = -\frac{1}{3} \\ x^0: & 6 = 12A & \Rightarrow A = 1/2 \end{array}$$

$$f(x) = \underline{\underline{\frac{1}{2x} + \frac{6x-1}{3(x^2+2)}}}$$

11. $f: y=\dfrac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}$

st $g_1 = 6$
st $g_2 = 5$ $\Big\}$ není ryzí

$$\begin{aligned}(x^6 - 2x^4 + 3x^3 - 9x^2 + 4)&:(x^5-5x^3+4x) = x + \frac{3x^4+3x^3-13x^2+4}{x^5-5x^3+4x}\\ -x^6 \pm 5x^4 \mp 4x^2 &\\ \hline 3x^4+3x^3-13x^2+4 &\end{aligned}$$

$$x^5-5x^3+4x = x(x^4-5x^2+4) = x(x^2-4)(x^2-1)$$
$$= x(x+2)(x-2)(x+1)(x-1)$$

$$f(x) = x + \boxed{\frac{3x^4+3x^3-13x^2+4}{x^5-5x^3+4x}} = x + \boxed{\frac{3x^4+3x^3-13x^2+4}{x(x+2)(x-2)(x+1)(x-1)}} =$$

$$= x + \boxed{\frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} + \frac{D}{x+1} + \frac{E}{x-1}} = \%$$

$$3x^4+3x^3-13x^2+4 = A(x+2)(x-2)(x+1)(x-1) + Bx(x-2)(x+1)(x-1) + Cx(x+2)(x+1)(x-1) +$$
$$+ Dx(x+2)(x-2)(x-1) + Ex(x+2)(x-2)(x+1)$$

$x=0$: $4 = 4A \Rightarrow A = 1$

$x=2$: $\underbrace{48+24-52+4}_{24} = 24C \Rightarrow C = 1$

$x=-2$: $\underbrace{48-24-52+4}_{-24} = 24B \Rightarrow B = -1$

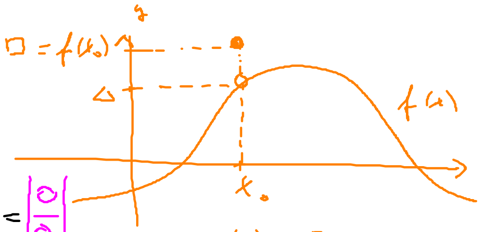
$x=1$: $-3 = -6E \Rightarrow E = \frac{1}{2}$

$x=-1$: $-9 = -6D \Rightarrow D = \frac{3}{2}$

$$\% = x + \frac{1}{x} - \frac{1}{x+2} + \frac{1}{x-2} + \frac{3}{2(x+1)} + \frac{1}{2(x-1)}$$

Příklad 10.1. Vypočitejte limity bez užití derivací:

1. $\lim\limits_{x\to -2^-} \dfrac{x^2-4}{|3x^2+5x-2|}$.



$$\lim_{x\to -2^-} \frac{x^2-4}{|3x^2+5x-2|} = \lim_{x\to -2^-} \frac{(x-2)(x+2)}{|(x+2)(3x-1)|} = \left|\frac{0}{0}\right|$$

$f(x_0) = \square$

$\lim\limits_{x\to x_0^-} f(x) = \lim\limits_{x\to x_0^+} f(x) = \triangle$

$\lim\limits_{x\to x_0} f(x) = \triangle$

$$= \lim_{x\to -2^-} \frac{(x-2)(x+2)}{|x+2|\cdot|3x-1|} = \lim_{x\to -2^-} \frac{(x-2)\cancel{(x+2)}}{-\cancel{(x+2)}|3x-1|} =$$

$$= -\lim_{x\to -2^-} \frac{x-2}{|3x-1|}$$ | je to spojitá fce tedy dosadím (-2) | $=$

$$= -\frac{-2-2}{|3(-2)-1|} = \underline{\underline{\frac{4}{7}}}$$

2. $\lim\limits_{x\to 1} \operatorname{tg}\left(\dfrac{2x^2-3x+1}{5x^2-6x+1}\pi\right)$.

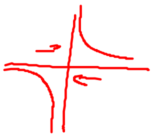
$$\lim_{x\to 1} \operatorname{tg}\left(\frac{2x^2-3x+1}{5x^2-6x+1}\pi\right) \left|\frac{0}{0}\right| = \lim_{x\to 1} \operatorname{tg}\left(\frac{(x-1)(2x-1)}{(x-1)(5x-1)}\pi\right) = \operatorname{tg}\left(\lim_{x\to 1} \frac{\cancel{(x-1)}(2x-1)}{\cancel{(x-1)}(5x-1)}\pi\right)$$

$$= \operatorname{tg}\left(\lim_{x\to 1} \frac{2x-1}{5x-1}\pi\right) = \operatorname{tg}\frac{1}{4}\pi = \underline{\underline{1}}$$

3. $\lim\limits_{x\to 1} \frac{x^3}{x^2-1}$.

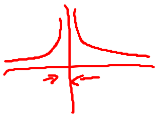
$\lim\limits_{x\to 1} \frac{x^3}{x^2-1} \quad \left|\frac{1}{0}\right|$ vyšetřujeme fci přes jednostranné limity

Pozn.: $\lim\limits_{x\to 0} \frac{1}{x} = \nexists$



$\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$
$\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$ } nerovnají se

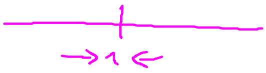
$\lim\limits_{x\to 0} \frac{1}{x^2} = \exists$



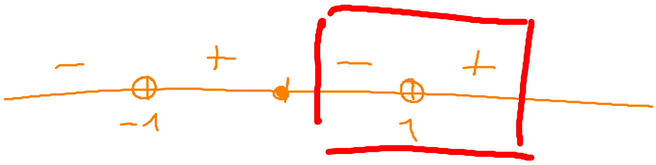
$\lim\limits_{x\to 0^+} \frac{1}{x^2} = \lim\limits_{x\to 0^-} \frac{1}{x^2} = \infty$

$\lim\limits_{x\to 1^+} \frac{x^3}{x^2-1} \quad \left|\frac{1}{0^+}\right| = \infty$

$\lim\limits_{x\to 1^-} \frac{x^3}{x^2-1} \quad \left|\frac{1}{0^-}\right| = -\infty$



$\Rightarrow \lim\limits_{x\to 1} \frac{x^3}{x^2-1} = \nexists$



5. $\lim\limits_{x\to-\infty} x\left(\sqrt{x^2+1}+x\right)$.

$$\lim_{x\to-\infty} x\left(\sqrt{x^2+1}+x\right) \quad \left|(-\infty)\cdot(\infty-\infty)\right| =$$

Pozn. $\lim\limits_{x\to\infty} x\left(\sqrt{x^2+1}+x\right) \quad \left|\infty\cdot\underbrace{(\infty+\infty)}_{\infty}\right|$

$\lim\limits_{x\to-\infty} x\left(\sqrt{x^2+1}-x\right) \quad \left|-\infty\underbrace{(\infty+\infty)}_{-\infty}\right|$ bez počítání

$$= \lim_{x\to-\infty} x\left(\sqrt{x^2+1}+x\right)\frac{\left(\sqrt{x^2+1}-x\right)}{\left(\sqrt{x^2+1}-x\right)} = \lim_{x\to-\infty}\frac{x\left(\cancel{x^2}+1-\cancel{x^2}\right)}{\sqrt{x^2+1}-x} \quad \left|\frac{-\infty}{\infty}\right| =$$

$$\sqrt{x^2+1} = \sqrt{x^2\left(1+\frac{1}{x^2}\right)} = |x|\sqrt{1+\frac{1}{x^2}}\,; \quad \sqrt{x^2} \neq x\ ! \quad \sqrt{x^2} = |x|\ !$$

$$= \lim_{x\to-\infty}\frac{x}{|x|\sqrt{1+\frac{1}{x^2}}-x} = \lim_{x\to-\infty}\frac{x}{-x\sqrt{1+\frac{1}{x^2}}-x} = -\lim_{x\to-\infty}\frac{\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{1}{x^2}}+1\right)} =$$

$$= -\frac{1}{2}$$

9. $\lim\limits_{x\to\infty} \operatorname{arctg}\frac{x^6+1}{x^6-x^2+3}$.

$$\lim_{x\to\infty} \operatorname{arctg}\frac{x^6+1}{x^6-x^2+3} \left|\frac{\infty}{\infty-\infty}\right| = \operatorname{arctg}\left(\lim_{x\to\infty}\frac{x^6+1}{x^6-x^2+3}\right) =$$

$$= \operatorname{arctg}\left(\lim_{x\to\infty}\frac{\cancel{x^6}\left(1+\overbrace{\tfrac{1}{x^6}}^{\to 0}\right)}{\cancel{x^6}\left(1-\underbrace{\tfrac{1}{x^4}}_{\to 0}+\underbrace{\tfrac{3}{x^6}}_{\to 0}\right)}\right) = \operatorname{arctg} 1 = \frac{\pi}{4}$$

10. $\lim_{x\to\infty} \operatorname{arccotg} \frac{x^4-1}{x^2+2}$.

$$\lim_{x\to\infty} \operatorname{arccotg} \frac{x^4-1}{x^2+2} \left|\frac{\infty}{\infty}\right| = \operatorname{arccotg}\left(\lim_{x\to\infty} \frac{x^{\cancel{4}\,2}\left(1-\frac{1}{x^4}\right)}{\cancel{x^2}\left(1+\frac{2}{x^2}\right)}\right) =$$

$$= \operatorname{arccotg} \infty = \underline{\underline{0}}$$

Příklady 11.1.

1. Užitím definice derivace určete f', jestliže

$$f:\ y = \frac{1}{\sqrt{x}},\ x > 0.$$

$$(x^m)' = m\,x^{m-1}$$

$$\left[\quad y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \qquad y' = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}} \quad\right]$$

Podle definice

$$y' = \lim_{x\to x_0} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x_0}}}{x - x_0} \left|\frac{0}{0}\right| = \lim_{x\to x_0} \frac{\frac{\sqrt{x_0} - \sqrt{x}}{\sqrt{x}\sqrt{x_0}}}{x - x_0} \cdot \frac{\sqrt{x_0} + \sqrt{x}}{\sqrt{x_0} + \sqrt{x}} =$$

$$= \lim_{x\to x_0} \frac{(x_0 - x)}{(x - x_0)\cdot\sqrt{x}\sqrt{x_0}\cdot(\sqrt{x_0} + \sqrt{x})} = -\lim_{x\to x_0} \frac{\cancel{(x_0 - x)}}{\cancel{(x_0 - x)}\sqrt{x}\sqrt{x_0}(\sqrt{x_0} + \sqrt{x})} =$$

$$= -\frac{1}{2x_0\sqrt{x_0}} = -\frac{1}{2\sqrt{x_0^3}} = -\frac{1}{2}x_0^{-\frac{3}{2}}$$

2. Určete $D(f)$, f', $D(f')$, jestliže

$$f: \quad y = \frac{5x^2}{\sqrt[5]{x^2}} + 30\sqrt[15]{x} + \frac{6}{\sqrt[3]{x}},\ x \neq 0.$$