Příklad 11.1.2. Určete $D(f)$, f', $D(f')$, jestliže

$$f : y = \frac{5x^2}{\sqrt[5]{x^2}} + 30\sqrt[15]{x} + \frac{6}{\sqrt[3]{x}}, x \neq 0.$$

$$(x^n)' = n \cdot x^{n-1}$$

$$f = 5x^{\frac{8}{5}} + 30x^{\frac{1}{15}} + 6x^{-\frac{1}{3}}$$

$$f' = \cancel{5} \cdot \frac{8}{\cancel{5}} x^{\frac{3}{5}} + \overset{2}{\cancel{30}} \cdot \frac{1}{\cancel{15}} x^{-\frac{14}{15}} + \overset{2}{\cancel{6}}\left(-\frac{1}{\cancel{3}}\right) x^{-\frac{4}{3}} = \underline{\underline{8\sqrt[5]{x^3} + 2\frac{1}{\sqrt[15]{x^{14}}} - 2\frac{1}{\sqrt[3]{x^4}}}}$$

$$D(f') = D(f) = \mathbb{R} - \{0\}$$

Příklad 11.1.3. Určete $D(f)$, f', $D(f')$, jestliže

$$f : y = \frac{2}{1-x^2}, |x| \neq 1. \qquad \mathbb{R} - \{-1, 1\}$$

$$f = 2(1-x^2)^{-1}$$

$$f' = 2(-1)(1-x^2)^{-2}(-2x) = \frac{4x}{(1-x^2)^2} \qquad D(f') = D(f)$$

$$f' = \frac{f_1}{f_2} = \frac{-2\cdot(-2x)}{(1-x^2)^2} = -\,''\,-$$

Příklad 11.1.4. Určete $D(f)$, f', $D(f')$, jestliže

$$f : y = e^x \left(1 + \operatorname{cotg} \frac{x}{2}\right).$$

$D(f) = \dots$ DÚ
$D(f') =$ -"-

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$f' = e^x \left(1 + \operatorname{cotg} \frac{x}{2}\right) + e^x \left(- \frac{1}{\sin^2 \frac{x}{2}} \cdot \frac{1}{2}\right) =$$

$$= e^x \left(1 + \operatorname{cotg} \frac{x}{2} - \frac{1}{2 \sin^2 \frac{x}{2}}\right)$$

Příklad 11.1.3. Určete $D(f)$, f', $D(f')$, jestliže

$$f : y = \underbrace{(\sin(\cos^2 x))}_{f_1} \cdot \underbrace{(\cos(\sin^2 x))}_{f_2}. \qquad D(f) = \mathbb{R}$$

$$f' = \cos(\cos^2 x) \cdot 2\cos x \cdot (-\sin x) \cdot (\cos(\sin^2 x)) +$$

$$+ (\sin(\cos^2 x))(-\sin(\sin^2 x)) \cdot 2\sin x \cdot \cos x = \%$$

$2\sin x \cos x = \sin 2x$ $\qquad$ $\cos^2 x - \sin^2 = \cos 2x$

$\cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta = \cos(\alpha - \beta)$

$$\% = -2\sin x \cos x \left(\cos(\cos^2 x) \cdot \cos(\sin^2 x) + \sin(\cos^2 x)\sin(\sin^2 x) \right) =$$

$$= -\sin 2x \cdot \cos(\cos^2 x - \sin^2 x) = -\sin 2x \cdot \cos(\cos 2x)$$

Příklad 11.2.1. Určete derivaci

$$f: y = \ln \underbrace{\sqrt[4]{\frac{1+x+x^2}{1-x+x^2}}}_{\triangle} + \frac{\sqrt{3}}{6} \operatorname{arctg} \frac{x\sqrt{3}}{1-x^2} + \sin\sqrt{\pi}.$$

$$\ln\frac{a}{b} = \ln a - \ln b$$

$$\ln a^n = n \ln a$$

$$\triangle = \frac{1}{4}\left(\ln(1+x+x^2) - \ln(1-x+x^2)\right)$$

$$f' = \frac{1}{\sqrt[4]{\frac{1+x+x^2}{1-x+x^2}}} \cdot \frac{1}{4}\left(\frac{1+x+x^2}{1-x+x^2}\right)^{-\frac{3}{4}} \cdot \frac{(1+2x)(1-x+x^2) - (1+x+x^2)(-1+2x)}{(1-x+x^2)^2} +$$

$$+ \ldots$$

$$f' = \frac{1}{4}\left(\frac{1+2x}{1+x+x^2} - \frac{-1+2x}{1-x+x^2}\right) + \frac{\sqrt{3}}{6}\,\frac{1}{1+\left(\frac{x\sqrt{3}}{1-x^2}\right)^2}\cdot\sqrt{3}\cdot\frac{1-x^2-x(-2x)}{(1-x^2)^2} =$$

$$= \frac{1}{4}\cdot\frac{(1+2x)(1-x+x^2)-(-1+2x)(1+x+x^2)}{(1+x+x^2)(1-x+x^2)} + \frac{\cancel{\sqrt{3}\sqrt{3}}\,(1+x^2)}{\cancel{6}\,\frac{(1-x^2)^2+3x^2}{\cancel{(1-x^2)^2}}\cdot\cancel{(1-x^2)^2}} =$$

$$= \frac{1}{\cancel{4}\,2}\,\frac{\cancel{2}(1-x^2)}{1+x^2+x^4} + \frac{1}{2}\,\frac{1+x^2}{1+x^2+x^4} =$$

$$(1+x+x^2)(1-x+x^2) = 1 \cancel{-x} \cancel{+x^2} \cancel{+x} \cancel{-x^2} \cancel{+x^3} + x^2 \cancel{-x^3} + x^4 = 1+x^2+x^4$$

$$(1+2x)(1-x+x^2)-(-1+2x)(1+x+x^2) = 1\cancel{-x}\cancel{+x^2}\cancel{+2x}-\cancel{2x^2}+\cancel{2x^3}+1\cancel{+x}\cancel{+x^2}\cancel{-2x} -$$

$$-2x^2-\cancel{2x^3} = 2 - 2x^2 = 2(1-x^2)$$

$$= \frac{\cancel{1}-\cancel{x^2}+\cancel{1}+\cancel{x^2}}{\cancel{2}(1+x^2+x^4)} = \frac{1}{1+x^2+x^4}$$

Příklad 11.2.2. Určete derivaci

$$f : y = \frac{1}{\sqrt{1+\cos x}} - \frac{1}{\sqrt{2}} \ln \sqrt{\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}-\sqrt{1+\cos x}}} + \cos^2\left(\frac{\pi}{2}-1\right).$$

$$f = (1+\cos x)^{-\frac{1}{2}} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\Big(\ln\left(\sqrt{2}+\sqrt{1+\cos x}\right) - \ln\left(\sqrt{2}-\sqrt{1+\cos x}\right)\Big) + \cos^2\left(\frac{\pi}{2}-1\right)$$

$$f' = \underbrace{-\frac{1}{2}(1+\cos x)^{-\frac{3}{2}}(-\sin x)}_{+\frac{\sin x}{2(1+\cos x)^{\frac{3}{2}}}} - \frac{1}{2\sqrt{2}}\left(\frac{\frac{1}{2}(1+\cos x)^{-\frac{1}{2}}\cdot(-\sin x)}{\sqrt{2}+\sqrt{1+\cos x}} - \frac{-\frac{1}{2}(1+\cos x)^{-\frac{1}{2}}(-\sin x)}{\sqrt{2}-\sqrt{1+\cos x}}\right)$$

$$= \dots \text{Dú} \dots = \frac{1}{\sqrt{1+\cos x}\cdot\sin x}$$

Příklad 11.3.2.b) Určete první a druhou derivaci $f'(x)$, $f''(x)$ a příslušné definiční obory funkcí,

$$f : y = \ln\sqrt{\frac{1-\sin x}{1+\sin x}}.$$

$$f = \frac{1}{2}\left(\ln(1-\sin x) - \ln(1+\sin x)\right)$$

$$f' = \frac{1}{2}\left(\frac{-\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x}\right) = -\frac{1}{2}\,\frac{\cos x(1+\sin x)+\cos x(1-\sin x)}{1-\sin^2 x} =$$

$$= -\frac{1}{\cancel{2}}\,\frac{\cancel{2}\,\cancel{\cos x}}{\cos^{\cancel{2}} x} = -\frac{1}{\cos x} = -(\cos x)^{-1}$$

$$f'' = -(-1)(\cos x)^{-2}\cdot(-\sin x) = -\frac{\sin x}{\cos^2 x}$$

Příklad 12.1.6. Dokažte, že tečny ke křivce $f : y = \dfrac{x-4}{x-2}$ v průsečících se souřadnými osami jsou rovnoběžné, určete rovnice tečny a normály v těchto bodech (načertněte obrazek).

$t: \quad y - f(x_0) = f'(x_0)(x - x_0) \qquad n: \quad y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0)$

$x = 0 \Rightarrow y = 2 \qquad B[0,2]$

$y = 0 \Rightarrow x = 4 \qquad A[4,0]$

$$y' = \frac{(x-2) - (x-4)}{(x-2)^2} = \frac{2}{(x-2)^2}$$

$$\left.\begin{array}{l} y'(A) = \frac{1}{2} = k_1 \\ y'(B) = \frac{1}{2} = k_2 \end{array}\right\} \Rightarrow \text{jsou rovnob.}$$

$A = [4,0]: \quad t_A: \; y - 0 = \frac{1}{2}(x-4)$, $\; y = \frac{1}{2}x - 2 \qquad n_A: \; y - 0 = -2(x-4)$, $\; y = -2x + 8$

$B = [0,2]: \quad t_B \; y - 2 = \frac{1}{2}x$, $\; y = \frac{1}{2}x + 2 \qquad n_B: \; y - 2 = -2x$, $\; y = -2x + 2$

Příklad 12.1.X. Určete tečny ke křivce $f : y = x^2 - 2x + 3$:

a) rovnoběžné s přímkou $p : 3x - y + 5 = 0$,

b) kolmé k přímce $q : x + y - 1 = 0$.

$t: \quad y - f(x_0) = f'(x_0)(x - x_0) \qquad f' = 2x - 2$

$n: \quad y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0)$

a) $p: 3x - y + 5 = 0 \Rightarrow k_p = 3 = k_t \Rightarrow 3 = 2x_0 - 2 \Rightarrow 2x_0 = 5$

$x_0 = \frac{5}{2}$

$f(x_0) = f\left(\frac{5}{2}\right) = \frac{25}{4} - \frac{10}{2} + 3 = \frac{17}{4} \Rightarrow A = \left[\frac{5}{2}, \frac{17}{4}\right]$

$t: \quad y - \frac{17}{4} = 3\left(x - \frac{5}{2}\right) \quad / \cdot 4$

$4y - 17 = 12x - 30$

$\underline{\underline{12x - 4y - 13 = 0}}$

$n: \quad y - \frac{17}{4} = -\frac{1}{3}\left(x - \frac{5}{2}\right) \quad / \cdot 12$

$12y - 51 = -4x + 10$

$\underline{\underline{4x + 12y - 61 = 0}}$

b) $q: \quad y = -x + 1 \Rightarrow k_q = -1 = k_n \quad \left(k_t = -\frac{1}{k_n}\right) \Rightarrow k_t = 1 \Rightarrow 1 = 2x_0 - 2$

$3 = 2x_0$

$\frac{3}{2} = x_0$

$f(x_0) = f\left(\frac{3}{2}\right) = \frac{9}{4} - 3 + 3 = \frac{9}{4} \Rightarrow B\left[\frac{3}{2}, \frac{9}{4}\right]$

$t: \quad y - \frac{9}{4} = 1\left(x - \frac{3}{2}\right) \quad / \cdot 4$

$4y - 9 = 4x - 6$

$\underline{\underline{4x - 4y + 3 = 0}}$

$n: \quad y - \frac{9}{4} = -1\left(x - \frac{3}{2}\right) \quad / \cdot 4$

$4y - 9 = -4x + 6$

$\underline{\underline{4x + 4y - 15 = 0}}$