Příklad 12.2.1. Vypočítejte s použitím L'Hospitalova pravidla:

$$\lim_{x\to 0} \frac{\sin 3x^2}{\ln\cos(2x^2-x)}.$$

$$\lim_{x\to 0} \frac{\sin 3x^2}{\ln(\cos(2x^2-x))} \left|\frac{0}{0}\right| \overset{LP}{=} \lim_{x\to 0} \frac{\cos 3x^2 \cdot 3 \cdot 2x}{\frac{-\sin(2x^2-x)\cdot(2\cdot 2x-1)}{\cos(2x^2-x)}} =$$

L'Hospitalovo pravidlo

$a \in \mathbb{R} \cup \{-\infty, \infty\}$, f, g – diferencovatelné

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \vee \lim_{x\to a} |g(x)| = \infty$

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

$$= \lim_{x\to 0} \frac{6x \cdot \cos(3x^2)\cdot \cos(2x^2-x)}{(1-4x)\cdot \sin(2x^2-x)} \left|\frac{0}{0}\right| \overset{*}{=}$$

$*$ $\lim_{x\to x_0} f(x) = r$, $\lim_{x\to x_0} g(x) = s$, $\lim_{x\to x_0} (f(x)\cdot g(x)) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) =$

$= r \cdot s$

$$= \lim_{x\to 0} \frac{6\cdot \cos(3x^2)\cos(2x^2-x)}{1-4x} \cdot \lim_{x\to 0} \frac{x}{\sin(2x^2-x)} \overset{LP}{=} 6 \cdot \lim_{x\to 0} \frac{1}{\cos(2x^2-x)(4x-1)} = \underline{\underline{-6}}$$

Příklad 12.2.X. Vypočítejte s použitím L'Hospitalova pravidla:

$$\lim_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right).$$

$\left(\frac{1}{x}\right)' = \left(x^{-1}\right)' =$

$\left(\frac{1}{x^2}\right)' = \left(x^{-2}\right)' = -2x^{-3}$

$$\lim_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right) \left|\infty\cdot 0\right| = \lim_{x\to\infty}\frac{\left(1-\cos\frac{1}{x}\right)}{\frac{1}{x^2}} \left|\frac{0}{0}\right| \overset{LP}{=}$$

$$= \lim_{x\to\infty}\frac{\sin\frac{1}{x}\cdot\left(\cancel{-}\cancel{\frac{1}{x^2}}\right)}{\cancel{-}\frac{2}{x^{\cancel{3}}}} = \lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{2}{x}} \left|\frac{0}{0}\right| \overset{LP}{=} \lim_{x\to\infty}\frac{\cos\frac{1}{x}\cdot\cancel{\left(-\frac{1}{x^2}\right)}}{2\cdot\cancel{\left(-\frac{1}{x^2}\right)}} = \underline{\underline{\frac{1}{2}}}$$

Příklad 13.2. Určete Taylorův polynom $T_n(f, A, \mathrm{d}x)$, $n \in \mathbb{N}$, $A = [-2]$, je-li $f : y = \ln(3 - 4x)$.

$$T_n(f, x_0, h) = f(x_0) + df(x_0, h) + \frac{d^2 f(x_0, h)}{2!} + \dots + \frac{d^n f(x_0, h)}{n!}$$

$$h = (x - x_0)$$

$$d^n f(x_0, h) = f^n(x_0) \cdot h^n = f^n(x_0) \cdot (x - x_0)^n$$

$$y = \ln(3 - 4x) \qquad y(-2) = \ln(11)$$

$$y' = \frac{-4}{3 - 4x} = -4(3 - 4x)^{-1}$$

$$y'' = -4 \cdot (-1)(3 - 4x)^{-2}(-4) = -4 \cdot 4(3 - 4x)^{-2} = -4^2(3 - 4x)^{-2} = -\frac{1 \cdot 4^2}{(3 - 4x)^2} \qquad f''(-2) = -\frac{4^2}{11^2}$$

$$y''' = -4^2(-2)(3 - 4x)^{-3}(-4) = -2 \cdot 4^3(3 - 4x)^{-3} = -\frac{1 \cdot 2 \cdot 4^3}{(3 - 4x)^3} \qquad f'''(-2) = -\frac{1 \cdot 2 \cdot 4^3}{11^3}$$

$$y^{IV} = -2 \cdot 4^3(-3)(3 - 4x)^{-4}(-4) = -2 \cdot 3 \cdot 4^4(3 - 4x)^{-4}$$

$$y^{V} = -2 \cdot 3 \cdot 4^4(-4)(3 - 4x)^{-5}(-4) = -2 \cdot 3 \cdot 4 \cdot 4^5(3 - 4x)^{-5} = -\frac{2 \cdot 3 \cdot 4 \cdot 4^5}{(3 - 4x)^5}$$

$$y^{VI} = -2 \cdot 3 \cdot 4 \cdot 4^5(-5)(3 - 4x)^{-6}(-4) = -2 \cdot 3 \cdot 4 \cdot 5 \cdot 4^6(3 - 4x)^{-6} = -\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 4^6}{(3 - 4x)^6}$$

$$\downarrow \vdots$$

$$y^n = -\frac{(n-1)!\, 4^n}{(3 - 4x)^n}$$

$$df(x_0, \underset{\substack{\downarrow \\ (x - x_0)}}{h}) = f'(x_0)(x - x_0) \quad \dots \quad d^n f(x_0, h) = f^n(x_0)(x - x_0)^n$$

$$df(-2, (x+2)) = f'(-2)(x+2) = -\frac{4}{11}(x+2)$$

$$d^2 f(-2, (x+2)) = f''(-2)(x+2)^2 = -\frac{4^2}{11^2}(x+2)^2$$

$$d^3 f(-2, (x+2)) = f'''(-2)(x+2)^3 = -\frac{1 \cdot 2 \cdot 4^3}{11^3}(x+2)^3$$

$$\vdots$$

$$d^n f(-2, (x+2)) = f^n(-2)(x+2)^n = -\frac{(n-1)! \cdot 4^n}{11^n}(x+2)^n$$

Příklad 13.2. Určete Taylorův polynom $T_n(f, A, \mathrm{d}x)$, $n \in \mathbb{N}$, $A = [-2]$, je-li $f : y = \ln(3 - 4x)$.

$$T_n = \ln 11 - \frac{4}{11}(x+2) - \frac{4^2}{2\, 11^2}(x+2)^2 - \underbrace{\frac{\cancel{2}\, 4^3}{\cancel{2}3\, 11^3}(x+2)^3}_{\frac{4^3}{3\cdot 11^3}(x+2)^3} - \dots - \underbrace{\frac{\cancel{(n-1)!}\, 4^n}{\cancel{n!}\, 11^n}(x+2)^n}_{\frac{4^n}{n\, 11^n}(x+2)^n}$$

$$= \ln 11 - \sum_{k=1}^{n} \frac{4^k}{k \cdot 11^k}(x+2)^k$$

$$T_3(-2, (x+2)) = \ln 11 - \frac{4}{11}(x+2) - \frac{4^2}{2\, 11^2}(x+2)^2 - \frac{\cancel{2}\, 4^3}{\cancel{2}3\, 4^3}(x+2)^3$$

Příklad 11.4.X. Určete Maclaurinův polynom $n-$tého stupně v bodě $x_0 = 0$ funkce

$$f : y = x \cdot e^x.$$

$x_0 = 0 \qquad (x - x_0) = x$

$d^n f(0,h) = f^n(0) \cdot x^n$

$y = x \cdot e^x$

$y' = e^x + xe^x$

$y'' = e^x + e^x + xe^x = 2e^x + xe^x$

$y''' = 2e^x + e^x + xe^x = 3e^x + xe^x$

$y^{IV} = 3e^x + e^x + xe^x = 4e^x + xe^x$

$\vdots$

$y^n = ne^x + xe^x$

$y(0) = 0$

$y'(0) = 1$

$y''(0) = 2$

$y'''(0) = 3$

$y^{IV}(0) = 4$

$\vdots$

$y^n(0) = n$

$$T_n(0) = M_n(0) = f(0) + \frac{f'(0)}{1!} \cdot x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^n(0)}{n!} \cdot x^n =$$

$$= 0 + \frac{1}{1!} x + \underset{\frac{1}{1!}}{\frac{2}{2!}} x^2 + \underset{\frac{1}{2!}}{\frac{3}{3!}} x^3 + \underset{\frac{1}{3!}\cdots}{\cdots} + \underset{\frac{1}{(n-1)!}}{\frac{n}{n!}} x^n$$

$$= \sum_{k=1}^{n} \frac{1}{(k-1)!} x^k$$

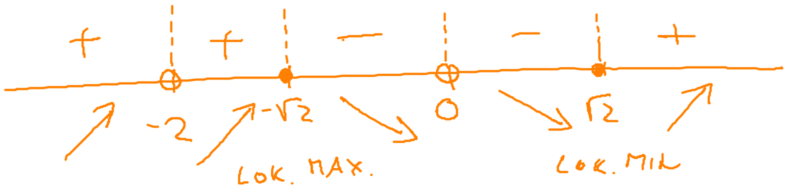
Příklad 11.6.j. Určete průběh funkce

$$f : y = \frac{e^x}{x^2 + 2x}.$$

① D_f: $x^2+2x \neq 0 \Rightarrow x(x+2) \neq \Rightarrow \begin{matrix} x \neq 0 \\ x \neq -2 \end{matrix}$ $\quad D(f) = \mathbb{R} - \{0, -2\}$

② $y' = \dfrac{e^x(x^2+2x) - e^x(2x+2)}{(x^2+2x)^2} = \dfrac{e^x(x^2+2x-2x-2)}{x^2(x+2)^2} = \dfrac{\overbrace{e^x}^{>0}(x^2-2)}{(x^2+2x)^2} = 0$

$\Rightarrow x = \pm\sqrt{2}$



$\left[-\sqrt{2}, \dfrac{e^{-\sqrt{2}}}{2-2\sqrt{2}}\right]$ – lok. max

$\left[\sqrt{2}, \dfrac{e^{\sqrt{2}}}{2+2\sqrt{2}}\right]$ – lok. min

③ $y'' = \dfrac{(e^x(x^2-2) + e^x 2x)\cdot(x^2+2x)^2 - e^x(x^2-2)\,2(x^2+2x)\cdot(2x+2)}{(x^2+2x)^4} =$

$= \dots = \dfrac{\overbrace{e^x}^{>0}\overbrace{(x^4-2x^2+4x+8)}^{>0}}{(x^2+2x)^3} = 0$ $\quad$ infl. body neexistují

$\rightarrow x^3(x+2)^3$

④ a) $\lim\limits_{x\to 0^+} \dfrac{e^x}{x^2+2x} = \dfrac{1}{0^+} = \infty$ $\qquad \lim\limits_{x\to 0^-} \dfrac{e^x}{x^2+2x} = \dfrac{1}{0^-} = -\infty$

$\lim\limits_{x\to -2^+} \dfrac{e^x}{\underbrace{x^2+2x}_{x(x+2)}} = \dfrac{k}{0^-} = -\infty$ $\qquad \lim\limits_{x\to -2^-} \dfrac{e^x}{x(x+2)} = \dfrac{k}{0^+} = \infty$

b) $y = kx + q$

k: $\lim\limits_{x\to\pm\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \dfrac{e^x}{\underbrace{x^2(x+2)}_{x^3+2x^2}} \left|\dfrac{\infty}{\infty}\right| \left|\dfrac{0}{\infty}\right| \stackrel{LP}{=} \lim\limits_{x\to\pm\infty} \dfrac{e^x}{3x^2+4x} \left|\dfrac{\infty}{\infty}\right| \left|\dfrac{0}{\infty}\right| \stackrel{LP}{=}$

$= \lim\limits_{x\to\pm\infty} \dfrac{e^x}{6x+4} \left|\dfrac{\infty}{\infty}\right| \left|\dfrac{0}{\infty}\right| \stackrel{LP}{=} \lim\limits_{x\to\pm\infty} \dfrac{e^x}{6} = \begin{cases} \infty & x\to\infty \text{ nemá (as.)} \\ 0 & x\to-\infty \;? \end{cases}$

q: $\lim\limits_{x\to-\infty} f(x) - kx = \lim\limits_{x\to-\infty} \dfrac{e^x}{x^2+2x} \left|\dfrac{0}{\infty}\right| \stackrel{LP}{=} \lim\limits_{x\to-\infty} \dfrac{e^x}{2x+2} \left|\dfrac{0}{\infty}\right| \stackrel{LP}{=}$

$= \lim\limits_{x\to-\infty} \dfrac{e^x}{2} = \underline{0}$ $\qquad x\to-\infty \quad \underline{\underline{y=0}}$

Příklad 11.6.j. Určete průběh funkce

$$f : y = \frac{e^x}{x^2 + 2x}.$$