Příklad 2.1

Vypočtěte inverzní matici k matici A metodou elementárních řádkových úprav a ověřte správnost výsledku, jestliže:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \begin{matrix} -2 \quad -1 \\ + \\ + \end{matrix} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \circlearrowleft \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -5 & -2 & 1 & 0 \end{array}\right) \overset{-1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & -1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{array}\right) \begin{matrix} \\ \sim \\ -2 \end{matrix}$$

$$\sim \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{E} \middle| \underbrace{\begin{matrix} \frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ -1 & 0 & 1 \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{matrix}}_{A^{-1}} \right) \qquad A^{-1} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ -1 & 0 & 1 \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 3 & 2 & -2 \\ -5 & 0 & 5 \\ 1 & -1 & 1 \end{pmatrix}$$

Příklad 2.2

Vypočtěte inverzní matici k matici A, jestliže:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 1 & 1 & -3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -3 & 0 & 0 & 1 \end{array}\right) \underset{+}{\overset{-2\;\;-1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ 0 & 1 & -5 & -1 & 0 & 1 \end{array}\right) \underset{+}{\overset{-1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -2 & 1 & 0 \\ \underline{0} & \underline{0} & \underline{0} & 1 & -1 & 1 \end{array}\right)$$

$h(A) = 2 \neq 3 = n \Rightarrow \nexists A^{-1}$

Příklad 2.3

Řešte maticovou rovnici $A \cdot X \cdot B = C$, jestliže:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & -1 \\ 2 & 4 \end{pmatrix}.$$

$$A \cdot X \cdot B = C \qquad / \cdot A^{-1}_{L}$$

$$\underbrace{A^{-1} \cdot A}_{E} \cdot X \cdot B = A^{-1} \cdot C \qquad / \cdot B^{-1}_{P}$$

$$X \cdot \underbrace{B \cdot B^{-1}}_{E} = A^{-1} \cdot C \cdot B^{-1}$$

$$X = A^{-1} \cdot C \cdot B^{-1}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right) \overset{-3}{\underset{+}{\sim}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -5 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{5} & -\frac{1}{5} \end{array}\right) \overset{5}{\underset{-2}{\sim}} \left(\begin{array}{cc|cc} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{E} & \underbrace{\begin{matrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{matrix}}_{A^{-1}} \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right) \sim \cdots\cdots \sim \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{E} & \underbrace{\begin{matrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{matrix}}_{B^{-1}} \end{array}\right)$$

$$A^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{pmatrix}$$

$$B^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

$$X = \frac{1}{5}\underline{\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 2 & 4 \end{pmatrix}} \cdot \frac{1}{5}\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 4 & 9 \\ -2 & -7 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 17 & -3 \\ -11 & -1 \end{pmatrix}$$

Příklad 2.4

Vypočtěte determinant

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ -1 & 0 & 2 \end{vmatrix}.$$

$$\left|\begin{array}{ccc} 1 & 2 & -1 \\ 0 & 3 & 1 \\ -1 & 0 & 2 \end{array}\right| \begin{array}{cc} 1 & 2 \\ 0 & 3 \\ -1 & 0 \end{array} = 1 \cdot 3 \cdot 2 + 2 \; 1 \cdot (-1) + (-1) \cdot \cancel{0} \cdot 0 - (-1) \cdot 3 \cdot (-1) - 1 \cdot \cancel{1} \cdot 0 - 2 \cdot \cancel{0} \cdot 2 = 6 - 2 - 3 = \underline{\underline{1}}$$

Příklad 2.5

Odvoďte vlastnost 5.

Příklad 2.6

Odvoďte vlastnost 7.

Příklad 2.7

Odvoďte vlastnost 8.

5. $\begin{vmatrix} k a_{11} & k a_{12} \\ a_{21} & a_{22} \end{vmatrix} = k a_{11} a_{22} - k a_{12} a_{21} = k(a_{11} a_{22} - a_{12} a_{21}) = k \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

7. $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $|B| = \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix}$, pak $|C| = \begin{vmatrix} a_{11}+b_{11} & a_{12} \\ a_{21}+b_{21} & a_{22} \end{vmatrix} = (a_{11}+b_{11}) a_{22} - a_{12}(a_{21}+b_{21}) =$

$= (a_{11} a_{22} - a_{12} a_{21}) + (b_{11} a_{22} - b_{21} a_{12}) = |A| + |B|$

8. $\begin{vmatrix} a_{11} & a_{12} \\ a_{21}+k a_{11} & a_{22}+k a_{12} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + k \underbrace{\begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix}}_{=0} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

Příklad 2.8

Pro determinant $|A|$ určete subtedetminant A_{23} a příslušný algebraický doplněk $\overline{A}_{23}$, jestliže

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

$$A_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \qquad \overline{A}_{23} = (-1)^{2+3} \cdot A_{23}$$

Příklad 2.9

Vypočtěte determinant pomocí Laplaceova rozvoje, jestliže

$$\begin{vmatrix} 3 & 1 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ 4 & 2 & -3 & 5 \\ 6 & 1 & -2 & 0 \end{vmatrix}.$$

$$\underset{\substack{-2 \qquad + \\ -1}}{\begin{vmatrix} 3 & \boxed{1} & 1 & -1 \\ 2 & 1 & 0 & 1 \\ 4 & 2 & -3 & 5 \\ 6 & 1 & -2 & 0 \end{vmatrix}} = \begin{vmatrix} 1 & 1 & 1 & -2 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 2 & -3 & 3 \\ 4 & 1 & -2 & -1 \end{vmatrix} \xleftarrow{LR} = \underbrace{0 \cdot (-1)^{2+1} \cdot \begin{vmatrix} 1 & 1 & -2 \\ 2 & -3 & 3 \\ 1 & -2 & -1 \end{vmatrix}}_{=0} + 1 \cdot (-1)^{2+2} \cdot \begin{vmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 4 & -2 & -1 \end{vmatrix} + 0 \cdot \ldots + 0 \cdot \ldots = \underset{+}{\begin{vmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 4 & -2 & -1 \end{vmatrix}}$$

$$= \begin{vmatrix} 1 & -1 & -2 \\ \hline 0 & 0 & 3 \\ \hline 4 & -3 & -1 \end{vmatrix} \xleftarrow{LR} = 0 \cdots + 0 \cdots + 3 \cdot (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ 4 & -3 \end{vmatrix} = -3 \cdot (-3+4) = -3$$

Příklad 2.10

Vypočtěte determinant pomocí přechodu na schodovitý tvar, jestliže

$$\begin{vmatrix} 3 & 1 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ 4 & 2 & -3 & 5 \\ 6 & 1 & -2 & 0 \end{vmatrix}.$$

$$\begin{vmatrix} 3 & 1 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ 4 & 2 & -3 & 5 \\ 6 & 1 & -2 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 2 & 4 & -3 & 5 \\ 1 & 6 & -2 & 0 \end{vmatrix} \begin{matrix} -1 \;\; -2 \\ + \\ + \\ + \end{matrix} - \begin{vmatrix} 1 & 3 & 1 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & -2 & -5 & 7 \\ 0 & 3 & -3 & 1 \end{vmatrix} (-1) = \begin{vmatrix} 1 & 3 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & -5 & 7 \\ 0 & 3 & -3 & 1 \end{vmatrix} \begin{matrix} 2 \;\; -3 \\ + \\ + \end{matrix} \begin{vmatrix} 1 & 3 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & -6 & 7 \end{vmatrix} \begin{matrix} -2 \\ + \end{matrix} \begin{vmatrix} 1 & 3 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot (-3) \cdot 1 = -3$$