Příklad 6.1

Určete všechna vlastní čísla a vlastní vektory matice

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

$$(A-\lambda E) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$$

$$|A-\lambda E| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 0$$

$$1 - 2\lambda + \lambda^2 - 1 = 0$$

$$\lambda \cdot (\lambda - 2) = 0 \quad \Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = 2$$

$\lambda_1 = 0$: $(A-\lambda E) \cdot X = 0$

$$1 \cdot x_1 + 1\, x_2 = 0$$

$$\cancel{1\, x_1 + 1\, x_2 = 0}$$

$x_2 = t$ $x_1 + t = 0$, $x_1 = -t$

$(-t, t)^T$, $t \in \mathbb{R} - 0$

$\lambda_2 = 2$

$$-x_1 + x_2 = 0$$

$$x_1 - x_2 = 0$$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$-x_1 + x_2 = 0$

$x_2 = s$

$-x_1 + s = 0$

$x_1 = s$

$(s, s)^T$, $s \in \mathbb{R} - 0$

Příklad 6.2

Určete všechna vlastní čísla a vlastní vektory matice

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 3 & -2 & -3 \\ 1 & 1 & -2 \end{pmatrix}.$$

$|A - \lambda E| = 0 \qquad (A - \lambda E)$

$$\begin{vmatrix} 2-\lambda & 1 & -3 \\ 3 & -2-\lambda & -3 \\ 1 & 1 & -2-\lambda \end{vmatrix} \overset{-1}{=} \begin{vmatrix} 2-\lambda & 1 & -3 \\ 3 & -2-\lambda & -3 \\ -1+\lambda & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 & -3 \\ 3 & -2-\lambda & -3 \\ -1 & 0 & 1 \end{vmatrix} =$$

$$(1-\lambda)\begin{vmatrix} 2-\lambda & 1 & -1-\lambda \\ 3 & -2-\lambda & 0 \\ -1 & 0 & 0 \end{vmatrix} = (1-\lambda)\cdot(-1-\lambda)\cdot(-1)^{1+3}\cdot\begin{vmatrix} 3 & -2-\lambda \\ -1 & 0 \end{vmatrix} =$$

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$$= (1-\lambda)(-1-\lambda)(-2-\lambda) = 0 \quad \leadsto \quad (1-\lambda)(1+\lambda)(2+\lambda) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = -2$$

$\lambda_1 = 1$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 3 & -3 & -3 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right) \begin{matrix} (-3) \; -1 \end{matrix} \qquad \begin{aligned} x_1 + x_2 - 3x_3 &= 0 \\ 3x_1 - 3x_2 - 3x_3 &= 0 \\ x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{aligned} x_1 + x_2 - 3x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_3 &= t \\ x_2 &= t \end{aligned} \qquad \begin{aligned} x_1 + t - 3t &= 0 \\ x_1 &= 2t \end{aligned}$$

$(2t, t, t)^T, \; t \in \mathbb{R} - 0$

$\lambda_2 = -1$:

$$\left(\begin{array}{ccc|c} 3 & 1 & -3 & 0 \\ 3 & -1 & -3 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 3 & 1 & -3 & 0 \\ 3 & -1 & -3 & 0 \end{array}\right) \overset{-3}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ \cancel{0} & \cancel{-4} & \cancel{0} & \cancel{0} \end{array}\right)$$

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_2 &= 0 \end{aligned} \qquad x_2 = 0 \qquad x_1 - x_3 = 0 \qquad \begin{aligned} x_3 &= s \\ x_1 &= s \end{aligned}$$

$(s, 0, s)^T, \; s \in \mathbb{R} - 0$

$\lambda_3 = -2$ .... $(p, -p, p)^T, \; p \in \mathbb{R} - 0$

$$\left(\begin{array}{ccc|c} 4 & 1 & -3 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \overset{/:3}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 4 & 1 & -3 & 0 \end{array}\right) \overset{-1}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ \cancel{3} & \cancel{0} & \cancel{-3} & \cancel{0} \end{array}\right)$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_3 &= 0 \end{aligned} \qquad x_1 = p \qquad \left.\begin{aligned} x_2 &= -x_1 = -p \\ x_3 &= x_1 = p \end{aligned}\right\} \Rightarrow$$