Příklad 11.1

Odvoďte z definice derivaci $f'(x_0)$ funkce $f(x) = x^3$.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$\lim_{x \to x_0} \frac{x^3 - x_0^3}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{(x - x_0)}(x^2 + x x_0 + x_0^2)}{\cancel{x - x_0}} = \lim_{x \to x_0} (x^2 + x x_0 + x_0^2) =$$

$$= 3x_0^2$$

$$f(x) = x^n \longrightarrow f'(x) = n \cdot x^{n-1}$$

$$= x^3 \longrightarrow = 3x^2$$

Příklad 11.2

Napište rovnici tečny a normály pro $f(x) = x^3$ v bodě $A = [2, ?]$.

$t: \quad y - f(x_0) = f'(x_0)(x - x_0)$

$n: \quad y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0)$

$f(x_0) = f(2) = 2^3 = 8 \implies A[2, 8]$

$f'(x) = (x^3)' = 3x^2 \qquad f'(x_0) = f'(2) = 3 \cdot 2^2 = 12$

$t: \quad y - 8 = 12(x - 2)$

$0 = 12x - y - 24 + 8$

$\underline{\underline{0 = 12x - y - 16}}$

$n: \quad y - 8 = -\frac{1}{12}(x - 2) \quad / 12$

$12y - 96 = -x + 2$

$\underline{\underline{x + 12y - 98 = 0}}$

$[(12, -1) \cdot (1, 12) = 12 + (-12) = 0]$

Příklad 11.3

Zadané funkce derivujte a výsledek upravte

a) $f(x) = \arccos x$,

b) $f(x) = \ln^2 x$,

c) $f(x) = \ln(x^2)$,

d) $f(x) = \sin(x^2)$,

e) $f(x) = \sin^2 x$,

f) $f(x) = x^{x^2}$,

g) $f(x) = -\dfrac{\cos x}{2\sin^2 x} + \ln\sqrt{\dfrac{1+\cos x}{\sin x}}$.

$\sin^2 y + \cos^2 y = 1$

a) $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$ $(f^{-1})'(x_0) = \dfrac{1}{f'(y_0)}$ $x_0 = f(y_0)$

$$(\arccos x)' = \frac{1}{(\cos y)'} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-\cos^2(\arccos x)}} = -\frac{1}{\sqrt{1-x^2}}$$

$y = \arccos x$

$\cos y = x$

b) $y = \ln^2 x$

$$y' = 2\ln x \cdot \frac{1}{x} = \underline{\underline{\frac{2\ln x}{x}}}$$

c) $y = \ln(x^2)$

$$y' = \frac{1}{\cancel{x^2}} \cdot 2\cancel{x} = \underline{\underline{\frac{2}{x}}}$$

d) $y = \sin(x^2)$

$$y' = \cos x^2 \cdot 2x = \underline{\underline{2x\cos x^2}}$$

e) $y = \sin^2 x = (\sin x)^2 = \sin x \cdot \sin x$

$$y' = 2\sin x \cdot \cos x = \underline{\underline{\sin 2x}}$$

$$y' = \cos x \cdot \sin x + \sin x \cos x = 2\sin x \cos x = \underline{\underline{\sin 2x}}$$

f) $y = x^{x^2} \left[f(x)^{g(x)} = e^{g(x) \cdot \ln f(x)} \right] = e^{x^2 \ln x}$

$$y' = e^{x^2 \cdot \ln x} \cdot \left(2x \cdot \ln x + \cancel{x^2} \cdot \frac{1}{\cancel{x}}\right) = e^{x^2 \ln x}\left(2x \ln x + x\right) =$$

$$= e^{x^2 \ln x}\, x\,(2\ln x + 1) = x^{x^2} \cdot x\,(2\ln x + 1) = \underline{\underline{x^{x^2+1}(2\ln x + 1)}}$$

g) $y = -\dfrac{\cos x}{2\sin^2 x} + \ln\sqrt{\dfrac{1+\cos x}{\sin x}}$

$$= -\frac{\cos x}{2\sin^2 x} + \ln\left(\frac{1+\cos x}{\sin x}\right)^{\frac{1}{2}} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left(\frac{1+\cos x}{\sin x}\right)$$

$$\boxed{y' = -\frac{-\sin x \cdot 2\sin^2 x - \cos x \cdot 2 \cdot 2 \cdot \sin x \cos x}{4 \cdot \sin^4 x} + \frac{1}{2} \cdot \frac{1}{\frac{1+\cos x}{\cancel{\sin x}}} \cdot \frac{-\sin x \cdot \sin x - (1+\cos x)\cos x}{\sin^{\cancel{2}} x}} =$$

$$= \frac{1}{2} \cdot \frac{\sin^2 x + 2\cos^2 x}{\sin^3 x} + \frac{1}{2} \cdot \frac{-\sin^2 x - \cos x - \cos^2 x}{(1+\cos x)\sin x} =$$

$$= \frac{1}{2}\,\frac{\sin^2 x + 2\cos^2 x}{\sin^3 x} + \frac{1}{2} \cdot \frac{-\underbrace{(\sin^2 x + \cos^2 x)}_{=1} - \cos x}{(1+\cos x)\sin x} =$$

$$= \frac{1}{2}\,\frac{\sin^2 x + 2\cos^2 x}{\sin^3 x} - \frac{1}{2}\,\frac{\cancel{1+\cos x}}{\cancel{(1+\cos x)}\sin x} \cdot \frac{\sin^2 x}{\sin^2 x} =$$

$$= \frac{\cancel{\sin^2 x} + \cancel{2}\cos^2 x - \cancel{\sin^2 x}}{\cancel{2}\sin^3 x} = \boxed{\frac{\cos^2 x}{\sin^3 x}}$$