FAKULTA STAVEBNÍ ústav matematiky a deskriptivní geometrie

BAA009 Matematika II (G)

Cvičení č. 2

Příklad 2.1 ~~$\displaystyle\int \frac{1}{2x^2+9x-5}\,\mathrm{d}x$~~

1.12) $\displaystyle\int e^x \cdot \sin x \, dx \left| \begin{matrix} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x \end{matrix} \right| = -e^x \cos x + \int e^x \cdot \cos x \, dx$

$\left| \begin{matrix} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x \end{matrix} \right| = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

$$\int e^x \sin x \, dx = e^x(\sin x - \cos x) - \int e^x \sin x \, dx$$

$$2\int e^x \sin x \, dx = e^x(\sin x - \cos x)$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x)$$

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Příklad 2.1 $\displaystyle\int \frac{1}{2x^2+9x-5}\,\mathrm{d}x$

$$\int \frac{1}{\underbrace{2x^2+9x-5}_{x_{1,2} = \frac{1}{2},\ -5}}\,dx = \int \frac{1}{(x+5)(2x-1)}\,dx \overset{*}{=} -\frac{1}{11}\int \frac{1}{x+5}\,dx + \frac{2}{11}\int \frac{1}{2x-1}\,dx =$$

$\int \frac{1}{\tilde{x}}$, $\tilde{x} = ax+b$

$$* \quad \frac{1}{(x+5)(2x-1)} = \frac{A}{x+5} + \frac{B}{2x-1} = \frac{-\frac{1}{11}}{x+5} + \frac{\frac{2}{11}}{2x-1}$$

$$\Rightarrow \quad 1 = A(2x-1) + B(x+5)$$

$x=-5: \quad 1 = -11A \Rightarrow A = -\frac{1}{11}$

$x=\frac{1}{2}: \quad 1 = \frac{11}{2}B \Rightarrow B = \frac{2}{11}$

$$* \quad \int f(ax+b) = \frac{F(ax+b)}{a}$$

$$= -\frac{1}{11}\ln|x+5| + \frac{2}{11}\,\frac{\ln|2x-1|}{2} + c = -\frac{1}{11}\ln|x+5| + \frac{1}{11}\ln|2x-1| + c =$$

$$= \frac{1}{11}\ln\left|\frac{2x-1}{x+5}\right| + c$$

$$\int f(ax+b) = \frac{F(ax+b)}{a}$$

Příklad 2.2 $\displaystyle\int \frac{9x-5}{9x^2-6x+1}\,\mathrm{d}x$

$$\int \frac{9x-5}{9x^2-6x+1}\,dx = \int \frac{9x-5}{(3x-1)^2}\,dx \overset{*}{=} \int \frac{3(3x-1)-2}{(3x-1)^2}\,dx = 3\int \frac{\cancel{3x-1}}{(3x-1)^{\cancel{2}}} - 2\int \frac{1}{(3x-1)^2} = 3\int \frac{1}{3x-1}\,dx - 2\int \frac{1}{(3x-1)^2}\,dx \left| \begin{array}{l} t = 3x-1 \\ dt = 3\,dx \Rightarrow dx = \frac{1}{3}\,dt \end{array} \right| = 3\frac{1}{3}\int \frac{1}{t}\,dt - 2\frac{1}{3}\int \frac{1}{t^2}\,dt = \ln|t| - \frac{2}{3}\frac{t^{-1}}{-1} = \ln|t| + \frac{2}{3t} + C \overset{*}{=}$$

$*$ LZE I POMOCÍ ROZKLADU NA PARCIÁLNÍ ZLOMKY

$$= 3\cdot\frac{\ln|3x-1|}{3} - 2\cdot\frac{(3x-1)^{-1}}{-1\cdot 3} \overset{*}{=} \ln|3x-1| + \frac{2}{3(3x-1)} + C$$

Příklad 2.3 $\int \frac{3x+1}{x^2+2x+5}\,\mathrm{d}x$

$$\int \frac{3x+1}{x^2+2x+5}\,dx = \int \frac{\frac{3}{2}(2x+2)-2}{x^2+2x+5}\,dx = \frac{3}{2}\underbrace{\int \frac{2x+2}{x^2+2x+5}\,dx}_{A} - 2\underbrace{\int \frac{1}{x^2+2x+5}\,dx}_{B} = \blacktriangle$$

$D<0$

$$A = \int \frac{2x+2}{x^2+2x+5}\,dx \left| \begin{array}{l} t = x^2+2x+5 \\ dt = (2x+2)\,dx \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| = \ln|x^2+2x+5|$$

$$\Longrightarrow \boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|}$$

$$B = \int \frac{1}{x^2+2x+5}\,dx \; \left\{ \int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x \right\} = \int \frac{1}{(x+1)^2-1+5}\,dx = \int \frac{1}{4+(x+1)^2}\,dx =$$

$$= \frac{1}{4}\int \frac{1}{1+\frac{1}{4}(x+1)^2}\,dx = \frac{1}{4}\int \frac{1}{1+\left[\frac{1}{2}(x+1)\right]^2}\,dx \left| \begin{array}{l} t = \frac{1}{2}(x+1) \\ dt = \frac{1}{2}\,dx \Rightarrow dx = 2\,dt \end{array} \right| =$$

$$= \frac{1}{4}\cdot 2\int \frac{1}{1+t^2}\,dt = \frac{1}{2}\operatorname{arctg} t \overset{*}{=}$$

$$= \frac{1}{4}\cdot 2\cdot \operatorname{arctg}\frac{x+1}{2} \overset{*}{=} \frac{1}{2}\operatorname{arctg}\frac{x+1}{2}$$

$$\blacktriangle = \underline{\underline{\frac{3}{2}\cdot \ln(x^2+2x+5) - \cancel{2}\,\frac{1}{\cancel{2}}\operatorname{arctg}\frac{x+1}{2} + C}}$$

Příklad 2.4 * $\displaystyle\int \frac{x^2-2x-7}{(x^2-2x+1)(x^2+2x+5)}\,\mathrm{d}x$

$$\left[\frac{1}{2}\ln|x-1|+\frac{1}{x-1}-\frac{1}{4}\ln(x^2+2x+5)+\frac{1}{2}\operatorname{arctg}\frac{x+1}{2}+C\right]$$

* Úlohu vypočítejte a odevzdejte na samostatném papíře do modulu Moodle.

Příklad 2.5 $\displaystyle\int \frac{1}{1+x^3}\,\mathrm{d}x$

$$\int \frac{1}{1+x^3}\,dx = \int \frac{1}{(1+x)(x^2-x+1)}\,dx = \%$$

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}$$

$$\Rightarrow 1 = Ax^2 - Ax + A + Bx^2 + Cx + Bx + C$$

$$\begin{array}{ll|l} x^2: & 0 = A + B & B = -A \\ x^1: & 0 = -A + B + C & 0 = -A - A + 1 - A \Rightarrow 3A = 1 \Rightarrow A = \frac{1}{3} \\ x^0: & 1 = A + C & C = 1 - A \end{array} \qquad \left.\begin{array}{l} B = -\frac{1}{3} \\ A = \frac{1}{3} \\ C = \frac{2}{3} \end{array}\right\} *$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right) \sim \text{G.E.M} \Rightarrow *$$

$$\% = \frac{1}{3}\underbrace{\int \frac{1}{x+1}\,dx}_{R} - \frac{1}{3}\underbrace{\int \frac{x-2}{x^2-x+1}\,dx}_{S} = *$$

$$R = \int \frac{1}{x+1}\,dx = \ln|x+1|$$

$$S = \int \frac{x-2}{x^2-x+1}\,dx = \int \frac{\frac{1}{2}(2x-1) - \frac{3}{2}}{x^2-x+1}\,dx = \frac{1}{2}\underbrace{\int \frac{2x-1}{x^2-x+1}\,dx}_{T} - \frac{3}{2}\underbrace{\int \frac{1}{x^2-x+1}\,dx}_{U}$$

$$T = \int \frac{2x-1}{x^2-x+1}\,dx \quad \left| \begin{array}{l} t = x^2-x+1 \\ dt = (2x-1)\,dx \end{array} \right| \ldots\ldots = \ln|x^2-x+1|$$

$$U = \int \frac{1}{x^2-x+1}\,dx = \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx = \frac{4}{3}\int \frac{1}{1+\underbrace{\frac{4}{3}\left(x-\frac{1}{2}\right)^2}_{\underbrace{\left(\frac{2}{\sqrt{3}}\right)^2\left(x-\frac{1}{2}\right)^2}_{\left(\frac{2x-1}{\sqrt{3}}\right)^2}}}\,dx = \frac{4}{3}\int \frac{1}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2}\,dx =$$

$$\left\{ \int f(ax+b)\,dx = \frac{F(ax+b)}{a} \right\}$$

$$= \frac{4}{3} \cdot \frac{\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)}{\frac{2}{\sqrt{3}}} = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)$$

$$* = \frac{1}{3}\ln|x+1| - \frac{1}{3}\left(\frac{1}{2}\ln(x^2-x+1) - \frac{3}{2}\,\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)\right) + C =$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

Příklad 2.6 $\int \frac{1}{\sin x}\,\mathrm{d}x$

INTEGRACE GONIOMETRICKÝCH FUNKCÍ

$$\int R(\sin x, \cos x)\,dx, \quad \sin x = u, \ \cos x = v$$

1) $R(-u, v) = -R(u, v) \Rightarrow t = \cos x$ (JE LICHÁ VZHLEDEM K $\sin x$)
2) $R(u, -v) = -R(u, v) \Rightarrow t = \sin x$ (JE LICHÁ VZHLEDEM K $\cos x$)
3) $R(-u, -v) = R(u, v) \Rightarrow t = \operatorname{tg} x$ (JE SUDÁ K OBĚMA)
4) V OSTATNÍCH PŘÍPADECH $\Rightarrow t = \operatorname{tg}\frac{x}{2}$ UNIVERZÁLNÍ SUBSTITUCE

$$\int \frac{1}{\sin x}\,dx \ \left|\begin{array}{l} t = \cos x \\ dt = -\sin x\,dx \end{array}\right| = \int \frac{\sin x}{\sin^2 x}\,dx = \int \frac{\sin x}{1-\cos^2 x}\,dx = -\int \frac{1}{1-t^2}\,dt =$$

$$\int f(-x) = \int \frac{1}{-\sin x}\,dx = -\int \frac{1}{\sin x}\,dx = -\int f(x) \Rightarrow t = \cos x$$

$\sin^2 x + \cos^2 x = 1$

$$= \int \frac{1}{t^2-1}\,dt = \int \frac{1}{(t+1)(t-1)}\,dt = -\frac{1}{2}\int \frac{1}{t+1}\,dt + \frac{1}{2}\int \frac{1}{t-1}\,dt =$$

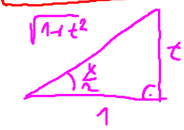
$$\frac{1}{t^2-1} = \frac{A}{t+1} + \frac{B}{t-1} \Rightarrow 1 = A(t-1) + B(t+1) \Rightarrow \begin{array}{l} 0 = A + B \\ 1 = -A + B \\ \hline 1 = 2B \end{array}$$

$$\Rightarrow B = \frac{1}{2}, \ A = -\frac{1}{2}$$

$$= -\frac{1}{2}\ln|t+1| + \frac{1}{2}\ln|t-1| = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

Příklad 2.7 $\displaystyle\int \frac{1-\sin x}{\cos x}\,\mathrm{d}x$

I) $\displaystyle\int \frac{1-\sin x}{\cos x}\,dx$ $\left|\begin{array}{l} t = tg\frac{x}{2} \Rightarrow arctg\, t = \frac{x}{2} \\ 2\,arctg\, t = x \\ \boxed{\frac{2}{1+t^2}\,dt = dx} \end{array}\right.$



$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$, $\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$

$\sin 2\alpha = 2\sin\alpha\cdot\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2} = \frac{2t}{1+t^2}$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$ $\Big| =$

$$= \int \frac{1-\frac{2t}{1+t^2}}{\frac{1-t^2}{\cancel{1+t^2}}}\,\frac{2}{\cancel{1+t^2}}\,dt = 2\int\frac{1+t^2-2t}{(1+t^2)(1-t^2)}\,dt = 2\int\frac{(t-1)^2}{(1+t^2)(1-t)(1+t)}\,dt = -2\int\frac{t-1}{(1+t^2)(1+t)}\,dt =$$

$$\frac{t-1}{(1+t^2)(1+t)} = \frac{A}{1+t} + \frac{Bt+C}{1+t^2}$$

$t-1 = A(1+t^2) + (Bt+C)(1+t)$

$t-1 = A + At^2 + Bt + C + Bt^2 + Ct$

$t^2:\ 0 = A + B \Rightarrow B = -A$

$t^1:\ 1 = B + C$ $\qquad 1 = -A + C$

$t^0:\ -1 = A + C$ $\qquad -1 = A + C$ $\qquad 2C = 0 \Rightarrow C = 0$, $A = -1$, $B = 1$

$$= -2\int\left(\frac{-1}{1+t} + \frac{t}{1+t^2}\right)dt = 2\int\frac{1}{1+t}\,dt - \int\frac{2t}{1+t^2}\,dt = 2\ln|1+t| - \ln|1+t^2| + c = \ln\left|\frac{(1+t)^2}{1+t^2}\right| + c$$

$\int\frac{f'(x)}{f(x)}\,dx$

$$= \ln\frac{(1+tg\frac{x}{2})^2}{1+tg^2\frac{x}{2}} + c$$

II) $\displaystyle\int\frac{1-\sin x}{\cos x}\,dx$ $\left|\begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array}\right| = \displaystyle\int\frac{(1-\sin x)\cos x}{\underbrace{\cos^2 x}_{1-\sin^2 x}}\,dx = \int\frac{1-t}{1-t^2}\,dt = \int\frac{\cancel{1-t}}{(\cancel{1-t})(1+t)}\,dt =$

$$= \int\frac{1}{1+t}\,dt = \ln|1+t| + c = \ln|1+\sin x| + c$$