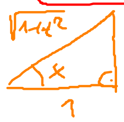
FAKULTA STAVEBNÍ ústav matematiky a deskriptivní geometrie

BAA009 Matematika II (G)

Cvičení č. 3

Příklad 3.1 $\displaystyle\int \frac{1}{\sin^2 x + 3\sin x\cos x + 2\cos^2 x}\,\mathrm{d}x$

$$\int \frac{1}{\sin^2 x + 3\sin x \cos x + 2\cos^2 x}\,dx \left| \begin{array}{l} \operatorname{tg} x = t \Rightarrow x = \operatorname{arctg} t \\ \boxed{dx = \frac{1}{1+t^2}\,dt} \end{array} \quad \begin{array}{l} \boxed{\sin x = \frac{t}{\sqrt{1+t^2}}} \\ \boxed{\cos x = \frac{1}{\sqrt{1+t^2}}} \end{array} \right| =$$



$$= \int \frac{1}{\frac{t^2}{1+t^2} + 3\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}} + 2\frac{1}{1+t^2}} \cdot \frac{1}{1+t^2}\,dt = \int \frac{1}{\frac{t^2+3t+2}{\cancel{1+t^2}}} \cdot \frac{1}{\cancel{1+t^2}}\,dt =$$

$$= \int \frac{1}{(t+1)(t+2)}\,dt \overset{*}{=} \int \frac{1}{t+1}\,dt - \int \frac{1}{t+2}\,dt = \ln|t+1| - \ln|t+2| + c = \ln\left|\frac{t+1}{t+2}\right| + c =$$

$$* \quad \frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2} \qquad 1 = A(t+2) + B(t+1) \quad \Rightarrow \quad A = 1,\ B = -1$$

$$\int f(ax+b) = \frac{F(ax+b)}{a}$$

$$= \underline{\underline{\ln\left|\frac{\operatorname{tg} x+1}{\operatorname{tg} x+2}\right| + c}}$$

Příklad 3.2 $\displaystyle\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}\,\mathrm{d}x$

$$\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}\,dx \;\left|\; \begin{array}{l} t=\sqrt[6]{x} \Rightarrow t^6 = x \\ 6t^5\,dt = dx \end{array} \right| = \int \frac{t^6+t^4+t}{t^{\cancel{6}}(1+t^2)}\,6\cancel{t^5}\,dt = 6\int \frac{\cancel{t}(t^5+t^3+1)}{\cancel{t}(1+t^2)}\,dt =$$

$$= 6\int \frac{t^3\cancel{(t^2+1)}}{\cancel{t^2+1}}\,dt + 6\int \frac{1}{1+t^2}\,dt = 6\,\frac{t^4}{4} + 6\,\mathrm{arctg}\,t + C = \underline{\underline{\frac{3}{2}\sqrt[6]{x^4} + 6\,\mathrm{arctg}\sqrt[6]{x} + C}}$$

Příklad 3.3 $\displaystyle\int \frac{1+\sqrt{\frac{x}{x+1}}}{x+1}\,\mathrm{d}x$

$$\int \frac{1+\sqrt{\frac{x}{x+1}}}{x+1}\,dx \quad \left|\; \begin{aligned} t &= \sqrt{\frac{x}{x+1}} \Rightarrow t^2 = \frac{x}{x+1} \\ xt^2 + t^2 &= x \\ xt^2 - x &= -t^2 \\ x(t^2-1) &= -t^2 \\ x &= \frac{t^2}{1-t^2} \end{aligned} \qquad \begin{aligned} dx &= \frac{2t(1-t^2) - t^2(-2t)}{(1-t^2)^2}\,dt \\ &= \frac{2t - 2t^3 + 2t^3}{(1-t^2)^2}\,dt \\ &= \frac{2t}{(1-t^2)^2}\,dt \end{aligned} \;\right| =$$

$$= \int \frac{1+t}{\frac{t^2}{1-t^2}+1} \cdot \frac{2t}{(1-t^2)^2}\,dt = \int \frac{(1+t)\,2t}{\frac{\cancel{t^2}+1-\cancel{t^2}}{\cancel{1-t^2}} \cdot (1-t^2)^{\cancel{2}}}\,dt = 2\int \frac{t\,\cancel{(1+t)}}{\cancel{(1+t)}(1-t)}\,dt = -2\int \frac{t-1+1}{t-1}\,dt =$$

$$= -2\int dt - 2\int \frac{1}{t-1}\,dt = -2t - 2\ln|t-1| + c = \underline{\underline{-2\sqrt{\frac{x}{x+1}} - 2\ln\left|\sqrt{\frac{x}{x+1}} - 1\right| + c}}$$

Příklad 3.4 $\displaystyle\int \sqrt{9-x^2}\,\mathrm{d}x$

$$\int\sqrt{9-x^2}\,dx = 3\int\sqrt{1-\left(\tfrac{x}{3}\right)^2}\,dx \left|\begin{array}{l} \frac{x}{3} = \sin t \Rightarrow \arcsin\frac{x}{3} = t \\ \frac{1}{3}dx = \cos t\,dt \Rightarrow dx = 3\cos t\,dt \end{array}\right| =$$

$$= 3\int\underbrace{\sqrt{1-\sin^2 t}}_{\cos^2 t}\,3\cos t\,dt = 9\int|\cos t|\cdot\cos t\,dt = 9\int\cos^2 t\,dt = 9\int\frac{1+\cos 2t}{2}\,dt =$$

$$= \frac{9}{2}\int dt + \frac{9}{2}\int\cos 2t\,dt = \frac{9}{2}t + \frac{9}{2}\cdot\frac{\sin 2t}{2} + c = \frac{9}{2}\left(t + \frac{1}{\cancel{2}}\cancel{2}\sin t\cos t\right) + c =$$

$$= \frac{9}{2}\left(t + \sin t\sqrt{1-\sin^2 t}\right) + c = \underline{\underline{\frac{9}{2}\left(\arcsin\frac{x}{3} + \frac{x}{3}\sqrt{1-\left(\tfrac{x}{3}\right)^2}\right) + c}}$$

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$$\int\sqrt{9-x^2}\,dx \left|\begin{array}{ll} u = \sqrt{9-x^2} & v' = 1 \\ u' = \frac{1}{2}(9-x^2)^{-\frac{1}{2}}(-2x) & v = x \end{array}\right| = \dots \quad x\sqrt{9-x^2} - \int\frac{-x^2}{\sqrt{9-x^2}}\,dx =$$

$$= x\sqrt{9-x^2} - \int\frac{\left(\sqrt{9-x^2}\right)^2 - 9}{\sqrt{9-x^2}}\,dx = x\sqrt{9-x} - \int\sqrt{9-x^2}\,dx + 9\int\frac{1}{\sqrt{9-x^2}}\,dx =$$

$$= \dots$$

Příklad 3.5 [*] $\displaystyle\int \frac{3x+2}{\sqrt{x^2+2x+3}}\,\mathrm{d}x$

$$\left[3\sqrt{x^2+2x+3} - \ln\left|x+1+\sqrt{x^2+2x+3}\right| + K\right]$$

[*] Úlohu vypočítejte a odevzdejte na samostatném papíře do modulu Moodle.

Příklad 3.6 $\displaystyle\int \frac{x-3}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$

$$\int \frac{x-3}{\sqrt{3-2x-x^2}}\,dx = \int \frac{-\frac{1}{2}(-2-2x)-4}{\sqrt{3-2x-x^2}}\,dx = -\frac{1}{2}\underbrace{\int \frac{-2-2x}{\sqrt{3-2x-x^2}}\,dx}_{A} - 4\underbrace{\int \frac{1}{\sqrt{3-2x-x^2}}\,dx}_{B} \overset{*}{=}$$

$$A: -\frac{1}{2}\int \frac{-2-2x}{\sqrt{3-2x-x^2}}\,dx \;\left|\begin{array}{l} t = 3-2x-x^2 \\ dt = -2-2x\,dx \end{array}\right| = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2}\int t^{-\frac{1}{2}}\,dt = -\frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} = -\sqrt{t} =$$

$$= -\sqrt{3-2x-x^2}$$

$$B: -4\int \frac{1}{\sqrt{3-2x-x^2}}\,dx = -4\int \frac{1}{\sqrt{3-(x^2+2x)}}\,dx = -4\int \frac{1}{\sqrt{3-(x+1)^2+1}}\,dx = -4\int \frac{1}{\sqrt{4-(x+1)^2}}\,dx$$

$$= -\frac{4}{2}\int \frac{1}{\sqrt{1-\left(\frac{x+1}{2}\right)^2}}\,dx \;\left|\begin{array}{l} t = \frac{1}{2}(x+1) \\ dt = \frac{1}{2}dx \Rightarrow 2dt = dx \end{array}\right| = -2\cdot 2\int \frac{1}{\sqrt{1-t^2}}\,dt = -4\arcsin t =$$

$$= -4\arcsin\frac{x+1}{2}$$

$$\overset{*}{=} \underline{\underline{-\sqrt{3-2x-x^2} - 4\arcsin\frac{x+1}{2} + c}}$$

Příklad 3.7 $\int \sqrt{x^2-4}\,\mathrm{d}x$

$$\int \sqrt{x^2-4}\,dx \quad \left|\begin{array}{ll} \sqrt{x^2-4} = t - x \quad /^2 & dx = \frac{2t^2-8}{4t^2}\,dt \\ \cancel{x^2} - 4 = t^2 - 2tx + \cancel{x^2} & dx = \frac{t^2-4}{2t^2}\,dt \\ 2tx = t^2+4 & x+\sqrt{x^2-4} = t \\ x = \frac{t^2+4}{2t} & \\ dx = \frac{2t\,2t - (t^2+4)2}{(2t)^2}\,dt & \end{array}\right| = \int \left(t - \frac{t^2+4}{2t}\right)\frac{t^2-4}{2t^2}\,dt =$$

$$= \int \frac{2t^2 - t^2 - 4}{2t}\cdot\frac{t^2-4}{2t^2}\,dt = \int \frac{(t^2-4)(t^2-4)}{4t^3}\,dt = \int \frac{t^4 - 8t^2 + 16}{4t^3}\,dt = \frac{1}{4}\int \frac{t^4}{t^3}\,dt - \frac{8}{4}\int \frac{t^2}{t^3} +$$

$$+ \frac{16}{4}\int \frac{1}{t^3}\,dt = \frac{1}{4}\int t\,dt - 2\int \frac{1}{t} + 4\int t^{-3}\,dt = \frac{1}{4}\cdot\frac{t^2}{2} - 2\ln|t| + 4\,\frac{t^{-2}}{-2} + c =$$

$$= \frac{t^2}{8} - 2\ln|t| - \frac{2}{t^2} + c = \underline{\underline{\frac{\left(x+\sqrt{x^2-4}\right)^2}{8} - 2\ln\left|x+\sqrt{x^2-4}\right| - \frac{2}{\left(x+\sqrt{x^2-4}\right)^2} + c}}$$