FAKULTA STAVEBNÍ ústav matematiky a deskriptivní geometrie

BAA009 Matematika II (G)

Cvičení č. 4

Příklad 4.1 $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$

NEWTONŮV INTEGRÁL

DEF: FCE F JE ZOBECNĚNÁ PRIMITIVNÍ FCE K FCI f NA (a,b):

a) F JE SPOJITÁ NA $\langle a,b\rangle$

b) $\exists F'(x) = f(x)$ $\forall x \in (a,b)$ „SKORO VŠUDE“

$$\rightarrow \int_a^b f(x)\,dx = [F(x)]_a^b = \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x)$$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = [\arcsin x]_{-1}^{1} = \lim_{x\to 1^-} \arcsin x - \lim_{x\to -1^+} \arcsin x = \arcsin 1 -$$

↳ JE SPOJITÁ NA $\langle -1,1\rangle$

$$- \arcsin(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \underline{\underline{\pi}}$$

Příklad 4.2 $\int_0^\infty \sin x \,\mathrm{d}x$

$$\int_0^\infty \sin x \,dx \;\Big|\; \text{sin MÁ NA } (0,\infty) \text{ PRIMITIVNÍ FUNKCI } -\cos x \text{ A PLATÍ...} \;\Big| = \left[-\cos x\right]_0^\infty = -\underbrace{\lim_{x\to\infty} \cos x}_{\lim_{x\to\infty}\cos x \ \nexists \lim} + \underbrace{\lim_{x\to 0^+} \cos x}_{\cos 0 = 1} =$$

$\Rightarrow \nexists$ ZADANÝ INTEGRÁL

Příklad 4.3 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+\cos^2 x}{\sin^2 x}\,\mathrm{d}x$

RIEMANŮV INTEGRÁL

F JE SPOJITÁ NA $\langle a, b\rangle$: $\displaystyle\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+\cos^2 x}{\sin^2 x}\,dx \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+1-\sin^2 x}{\sin^2 x}\,dx = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x}\,dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} dx = 2\left[-\mathrm{cotg}\, x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - [x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$

$$= -2\Big(\underset{\overset{\shortparallel}{0}}{\mathrm{cotg}\,\tfrac{\pi}{2}} - \underset{\overset{\shortparallel}{1}}{\mathrm{cotg}\,\tfrac{\pi}{4}}\Big) - \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \underline{\underline{2 - \frac{\pi}{4}}}$$

Příklad 4.4 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}\,\mathrm{d}x$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}\,dx \;\left|\; x\cdot\frac{1}{\sin^2 x} \quad \begin{array}{ll} u = x & v' = \frac{1}{\sin^2 x} \\ u' = 1 & v = -\operatorname{cotg} x \end{array}\right| = -\left[x\cdot\operatorname{cotg} x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cotg} x\,dx =$$

$$= -\left[x\operatorname{cotg} x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x}\,dx \left(\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|\right) = -\left[x\operatorname{cotg} x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} +$$

$$+\left[\ln|\sin x|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = -\left(\frac{\pi}{3}\cdot\operatorname{cotg}\frac{\pi}{3} - \frac{\pi}{4}\cdot\operatorname{cotg}\frac{\pi}{4}\right) + \left(\ln\left|\sin\frac{\pi}{3}\right| - \ln\left|\sin\frac{\pi}{4}\right|\right) =$$

$$= -\left(\frac{\pi}{3}\frac{\sqrt{3}}{3} - \frac{\pi}{4}\cdot 1\right) + \left(\ln\frac{\sqrt{3}}{2} - \ln\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} - \frac{\sqrt{3}}{9}\pi + \ln\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}} = \underline{\underline{\frac{\pi}{4} - \frac{\sqrt{3}}{9}\pi + \ln\frac{\sqrt{3}}{\sqrt{2}}}}$$

Příklad 4.5 $\displaystyle\int_1^e \frac{1+\ln x}{x}\,\mathrm{d}x$

$$\int_1^e \frac{1+\ln x}{x}\,dx \left| \begin{array}{ll} \ln x = t & \begin{array}{c|c|c} x & 1 & e \\ \hline t=\ln x & 0 & 1 \end{array} \\ \frac{1}{x}dx = dt & \end{array} \right| = \int_0^1 1+t \, dt \left[t + \frac{t^2}{2}\right]_0^1 =$$

$$= 1 + \frac{1}{2} - 0 - 0 = \underline{\underline{\frac{3}{2}}}$$

Příklad 4.6 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1-\sin^2 x}{\sin^3 x \cdot \cos x}\,\mathrm{d}x$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\overbrace{1-\sin^2 x}^{\cos^2 x}}{\sin^3 x \cdot \cos x}\,dx \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos^{\cancel{2}} x}{\sin^3 x\ \cancel{\cos x}}\,dx \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{\sin^3 x}\,dx \left| \begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array} \quad \begin{array}{c|c|c} x & \frac{\pi}{4} & \frac{\pi}{3} \\ \hline t & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} \end{array} \right|$$

$$= \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{t^3}\,dt = \left[\frac{t^{-2}}{-2}\right]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = -\frac{1}{2}\left[\frac{1}{t^2}\right]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = -\frac{1}{2}\left(\frac{1}{\frac{3}{4}} - \frac{1}{\frac{2}{4}}\right) = -\frac{1}{2}\left(\frac{4}{3} - 2\right) =$$

$$= -\frac{1}{2}\left(\frac{4-6}{3}\right) = \underline{\underline{\frac{1}{3}}}$$

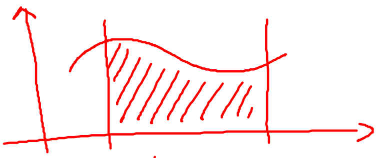
Příklad 4.7 $\displaystyle\int \frac{1}{1+\sin^2 x}\,\mathrm{d}x \longrightarrow \int_0^{\pi} \frac{1}{1+\sin^2 x}\,\mathrm{d}x$

$$\int \frac{1}{1+\sin^2 x}\,dx = \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\,\operatorname{tg} x) + C$$ DÚ ODVOĎTE

a) NEKOREKTNÍ POUŽITÍ ZÁKLADNÍ VĚTY INTEGRÁLNÍHO POČTU!

$$\int_0^{\pi} \frac{1}{1+\sin^2 x}\,dx \overset{*}{=} \left[\frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\,\operatorname{tg} x)\right]_0^{\pi} = \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\,\operatorname{tg}\pi) -$$
$$- \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\,\operatorname{tg} 0) = \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\cdot 0) - \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\cdot 0) = 0$$

POZOR! 1. ZADANÁ FUNKCE, KTEROU INTEGRUJEME JE KLADNÁ



2. U KLADNÉ FUNKCE URČITÝM INTEGRÁLEM POČÍTÁME PLOŠNÝ OBSAH, OHRANIČENÝ TOUTO FUNKCÍ A OSOU x

3. A TENTO PLOŠNÝ OBSAH NENÍ ROVEN NULE!

!! PRIMITIVNÍ FUNKCE NA UZAVŘENÉM INTERVALU $\langle a,b\rangle$ MUSÍ BÝT SPOJITÁ.

b) PRIMITIVNÍ FUNKCE JE NESPOJITÁ PRO $x=\frac{\pi}{2}$

$$\int_0^{\pi} \frac{1}{1+\sin^2 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2 x}\,dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{1+\sin^2 x}\,dx =$$

$$= \left[\lim_{x\to\frac{\pi}{2}^-} \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\,\operatorname{tg} x) - \lim_{x\to 0^+} \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\,\operatorname{tg} x)\right] +$$

$$\left[\lim_{x\to\pi^-} \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\,\operatorname{tg} x) - \lim_{x\to\frac{\pi}{2}^+} \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\,\operatorname{tg} x)\right] =$$

$$= \frac{1}{\sqrt{2}}\left(\lim_{a\to\infty}\operatorname{arctg} a - \operatorname{arctg} 0 + \operatorname{arctg} 0 - \lim_{a\to-\infty}\operatorname{arctg} a\right) =$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} - 0 + 0 - \left(-\frac{\pi}{2}\right)\right) = \underline{\underline{\frac{1}{\sqrt{2}}\pi}} \doteq \underline{\underline{2{,}22}}$$

Příklad 4.8 $\displaystyle\int_1^4 \frac{2\sqrt{x}}{5+\sqrt{x}}\,\mathrm{d}x$

$$\int_1^4 \frac{2\sqrt{x}}{5+\sqrt{x}}\,dx \quad \left| \begin{array}{l} t=\sqrt{x} \\ t^2 = x \\ 2t\,dt = dt \end{array} \quad \begin{array}{c|c|c} x & 1 & 4 \\ \hline t & 1 & 2 \end{array} \right| = \int_1^2 \frac{2t}{5+t}\cdot 2t\,dt = 4\int_1^2 \frac{t^2}{t+5}\,dt$$

$$= 4\int_1^2 \frac{t^2-25+25}{t+5}\,dt = 4\int_1^2 \frac{(t-5)\cancel{(t+5)}}{\cancel{t+5}}\,dt + 100\int \frac{1}{t+5}\,dt =$$

$$= 4\int_1^2 t\,dt - 20\int_1^2 dt + 100\int_1^2 \frac{1}{t+5}\,dt = 4\left[\frac{t^2}{2}\right]_1^2 - 20\left[t\right]_1^2 +$$

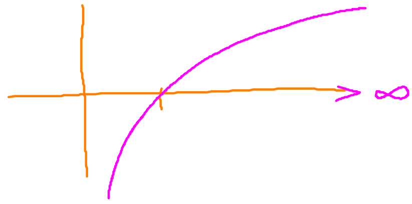
$$+ 100\left[\ln|t+5|\right]_1^2 = 4\left(\frac{4}{2}-\frac{1}{2}\right) - 20(2-1) + 100(\ln 7 - \ln 6) =$$

$$= \underline{\underline{100\ln\frac{7}{6} - 14}}$$

Příklad 4.9 $\int_{-\infty}^{\infty} \frac{\operatorname{arctg}^2 x}{1+x^2}\,\mathrm{d}x$

$$\int_{-\infty}^{\infty} \frac{\operatorname{arctg}^2 x}{1+x^2}\,dx \;\left|\; \begin{array}{l} \operatorname{arctg} x = t \\ \frac{1}{1+x^2}\,dx = dt \end{array} \quad \begin{array}{c|c|c} x & -\infty & \infty \\ \hline t & -\frac{\pi}{2} & \frac{\pi}{2} \end{array} \right| = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^2\,dt = \frac{1}{3}\left[t^3\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \frac{1}{3}\left(\frac{\pi^3}{8} + \frac{\pi^3}{8}\right) = \underline{\underline{\frac{\pi^3}{12}}}$$



Příklad 4.10 $\displaystyle\int_1^\infty \frac{x^3+1}{x^4}\,\mathrm{d}x$

$$\int_1^\infty \frac{x^3+1}{x^4}\,dx = \int_1^\infty \frac{1}{x} + \frac{1}{x^4}\,dx \left[\ln|x| - \frac{1}{3x^3}\right]_1^\infty = \lim_{x\to\infty}\left(\ln x - \frac{1}{3x^3}\right) -$$

$$-\left(\ln 1 - \frac{1}{3}\right) = \infty - 0 - 0 + \frac{1}{3} = \underline{\underline{\infty}}$$