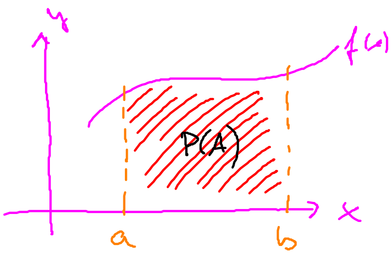
BAA009 Matematika II (G)

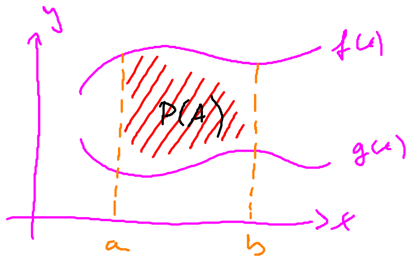
Cvičení č. 5

Příklad 5.1 Určete plošný obsah rovinného obrazce ohraničeného křivkami $y = 4 - x^2$, $y = 0$.



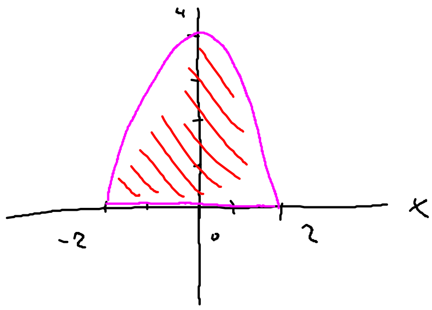
1. $y = f(x)$ $\qquad P(A) = \int_a^b |f(x)|\,dx$

2. $x = \varphi(t)$ $\quad t \in \langle \alpha, \beta \rangle$ $\qquad P(A) = \int_\alpha^\beta |\psi(t) \cdot \varphi'(t)|\,dt$
   $y = \psi(t)$



1. $P(A) = \int_a^b [f(x) - g(x)]\,dx$

2. $P(A) = \int_\alpha^\beta \psi(t)\,|\varphi'(t)|\,dt$



$y = 4 - x^2$
$y = 0$

$4 - x^2 = 0$
$(2-x)(2+x) = 0$
$x_1 = 2$
$x_2 = -2$

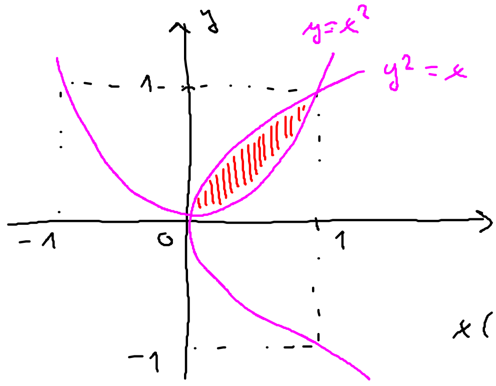
I $\quad P = \int_{-2}^{2} 4 - x^2\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 8 - \frac{8}{3} + 8 - \frac{8}{3} =$

$= 16 - \frac{16}{3} = \frac{32}{3}$

II $\quad$ SYMETRICKÁ OBLAST $\Rightarrow$ $P = 2\int_0^2 (4 - x^2)\,dx = 2\left[4x - \frac{x^3}{3}\right]_0^2 =$

$= 2\left(8 - \frac{8}{3} - (0 - 0)\right) = \frac{32}{3}$

Příklad 5.2 Určete plošný obsah rovinného obrazce ohraničeného křivkami $y = x^2$, $y^2 = x$.



$$\begin{aligned} y &= x^2 \\ y^2 &= x \\ \hline x^4 &= x \\ x^4 - x &= 0 \\ x(x^3-1) &= 0 \\ x(x-1)(x^2+x+1) &= 0 \\ x_1 &= 0 \\ x_2 &= 1 \end{aligned}$$

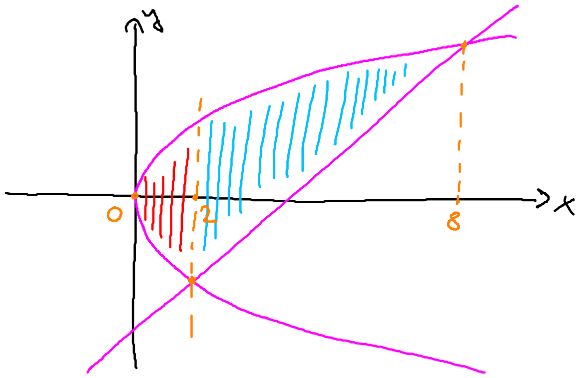
$$y^2 = x \rightarrow \begin{aligned} y_1 &= \sqrt{x} \\ y_2 &= -\sqrt{x} \end{aligned}$$

$$y = \pm\sqrt{x}$$

$$P = \int_0^1 \Big(\underset{\substack{\shortparallel \\ x^{\frac{1}{2}}}}{\sqrt{x}} - x^2 \Big)\, dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1 = \left[\frac{2}{3}\sqrt{x^3} - \frac{x^3}{3} \right]_0^1 = \frac{2}{3} - \frac{1}{3} - (0-0) =$$

$$= \underline{\underline{\frac{1}{3}}}$$

Příklad 5.3 Určete plošný obsah rovinného obrazce ohraničeného křivkami $y^2 = 2x$, $y = x - 4$.



$$\begin{aligned} y^2 &= 2x \\ y &= x-4 \\ \hline (x-4)^2 &= 2x \\ x^2-8x+16 &= 2x \\ x^2-10x+16 &= 0 \\ (x-2)(x-8) &= 0 \\ x_1 &= 2 \\ x_2 &= 8 \end{aligned}$$

$$y^2 = 2x$$
$$y = \pm\sqrt{2x}$$

$$P = P_1 + P_2 = \int_0^2 \sqrt{2x} - (-\sqrt{2x})\,dx + \int_2^8 \left(\sqrt{2x} - (x-4)\right) dx = \int_0^2 2\sqrt{2}\,\underset{x^{\frac{1}{2}}}{\underset{\shortparallel}{\sqrt{x}}}\,dx +$$

$$+ \int_2^8 \sqrt{2}\, x^{\frac{1}{2}} - x + 4\,dx = 2\sqrt{2}\left[\frac{2}{3}\sqrt{x^3}\right]_0^2 + \left[\sqrt{2}\,\frac{2}{3}\sqrt{x^3} - \frac{x^2}{2} + 4x\right]_2^8 =$$

$$= 2\sqrt{2}\,\frac{2}{3}\sqrt{8} - 0 + \left(\frac{\sqrt{2}\cdot 2}{3}\sqrt{8^3} - \frac{64}{2} + 32 - \left(\frac{2\sqrt{2}}{3}\sqrt{2^3} - \frac{4}{2} + 8\right)\right) =$$

$$= \cdots\cdots = \frac{16}{3} + \frac{38}{3} = \frac{54}{3} = \underline{\underline{18}}$$

Příklad 5.3 Určete plošný obsah rovinného obrazce daného parametricky $x = a\cos t$, $y = b\sin t$, $t \in \langle 0, 2\pi \rangle$, $a > 0$, $b > 0$.

– ELIPSA ; $P_{ELIPSA} = \pi a b$

$\varphi = a\cos t \Rightarrow \varphi' = -a\sin t$

$\psi = b\sin t$

$$P = \int_0^{2\pi} b\sin t \,|-a\sin t|\, dt = ab\int_0^{2\pi} \sin^2 t\, dt \overset{1.}{=} ab\int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt =$$

1. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$, $\sin^2\alpha + \cos^2\alpha = 1$ $\Big\}$ $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$, $\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$

2. $\operatorname{tg} t = s$

3. $\operatorname{tg}\frac{t}{2} = s$

$$\int f(ax-b)\,dx = \frac{F(ax-b)}{a}$$

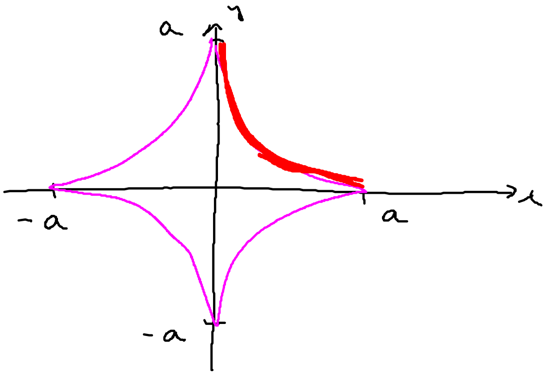
$$= \frac{ab}{2}\int_0^{2\pi} 1 - \cos 2t\, dt = \frac{ab}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} = \frac{ab}{2}2\pi - 0 = \underline{\underline{\pi a b}}$$

Příklad 5.5 Určete délku křivky dané parametricky $x = a\cos^3 t$, $y = a\sin^3 t$, $t \in \left\langle 0, \frac{\pi}{2}\right\rangle$, $A > 0$.

DÉLKA KŘIVKY:

1. $y = f(x)$, $x \in \langle a, b\rangle$ $\qquad L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

2. $x = \varphi(t)$, $y = \psi(t)$ $\quad t \in \langle \alpha, \beta\rangle$ $\qquad L = \int_\alpha^\beta \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2}\, dt$



ASTEROIDA

$x = a\cos^3 t \qquad t \in \langle 0, 2\pi\rangle$

$y = a\sin^3 t$

$x' = \varphi'(t) = a\, 3\cos^2 t \cdot (-\sin t) = -3a\cos^2 t \sin t$

$y' = \psi'(t) = a \cdot 3\sin^2 t \cdot \cos t = 3a\sin^2 t \cos t$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{(-3a\cos^2 t \sin t)^2 + (3a\sin^2 t\cos t)^2}\, dt =$$

$2\sin t\cos t = \sin 2t$

$\int f(ax+b)\,dx = \frac{F(ax+b)}{a}$

$$= \int_0^{\frac{\pi}{2}} \sqrt{3^2a^2\cos^4 t\sin^2 t + 3^2a^2\sin^4 t\cos^2 t}\, dt =$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{3^2a^2\cos^2 t\sin^2 t\,\underbrace{(\cos^2 t + \sin^2 t)}_{1}}\, dt = \frac{3a}{2}\int_0^{\frac{\pi}{2}} 2\sin t\cos t\, dt =$$

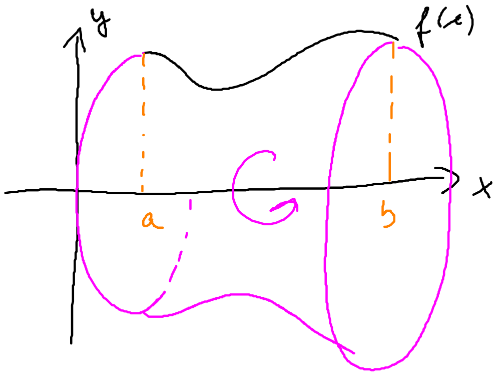
$$= \frac{3}{2}a\int_0^{\frac{\pi}{2}} \sin 2t\, dt = \frac{3}{2}a\left[-\frac{\cos 2t}{2}\right]_0^{\frac{\pi}{2}} = -\frac{3}{4}a(\cos\pi - \cos 0) =$$

$$= -\frac{3}{4}a(-1-1) = \frac{3}{2}a$$

CELÁ ASTEROIDA: $4L = 4\cdot\frac{3}{2}a = \underline{\underline{6a}}$

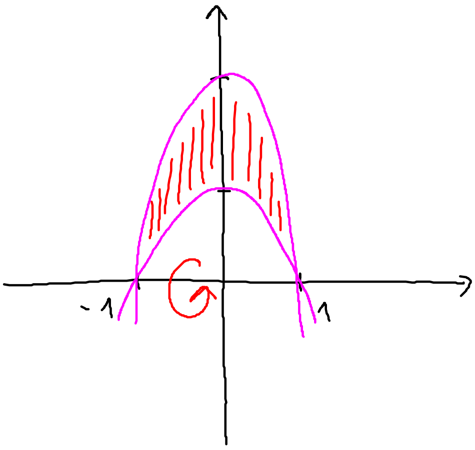
Příklad 5.6 Určete objem rotačního tělesa vzniklého rotací obrazce ohraničeného křivkami $y = -x^2 + 1$, $y = -2x^2 + 2$.

OBJEM ROTAČNÍHO TĚLESA



1. $y = f(x),\ x \in \langle a, b \rangle$ $\qquad V = \pi \int_a^b (f(x))^2\, dx$

2. $x = \varphi(t),\ t \in \langle \alpha, \beta \rangle$, $y = \psi(t)$ $\qquad V = \pi \int_\alpha^\beta \psi^2(t)\, |\varphi'(t)|\, dt$



$$y = -x^2 + 1$$
$$y = -2x^2 + 2$$

$$-2x^2 + 2 = -x^2 + 1$$
$$0 = x^2 - 1$$
$$0 = (x+1)(x-1)$$

$$x_1 = 1$$
$$x_2 = -1$$

$$V = V_1 - V_2 = \pi \int_{-1}^{1} (-2x^2 + 2)^2\, dx - \pi \int_{-1}^{1} (-x^2 + 1)^2\, dx$$

$$= \pi \int_{-1}^{1} 4x^4 - 8x^2 + 4 - (x^4 - 2x^2 + 1)\, dx = \pi \int_{-1}^{1} 3x^4 - 6x^2 + 3\, dx =$$

$$= \pi \left[\frac{3}{5} x^5 - \frac{6}{3} x^3 + 3x \right]_{-1}^{1} = \pi \left(\frac{3}{5} - 2 + 3 - \left(-\frac{3}{5} + 2 - 3 \right) \right) =$$

$$= \pi \left(\frac{6}{5} + 2 \right) = \underline{\underline{\frac{16}{5} \pi}}$$

Příklad 5.7 Určete povrch rotačního tělesa vzniklého rotací obrazce daného $-3 \le x \le 2$, $0 \le y \le \sqrt{4+x}$.

POVRCH PLÁŠTĚ ROTAČNÍHO TĚLESA

1. $$S = 2\pi \int_a^b |f(x)| \cdot \sqrt{1+[f'(x)]^2}\, dx$$

2. $$S = 2\pi \int_\alpha^\beta |\psi(t)| \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2}\, dt$$

$$f(x) = \sqrt{4+x} = (4+x)^{\frac{1}{2}}, \quad f'(x) = \frac{1}{2\sqrt{4+x}}$$

$$S = 2\pi \int_{-3}^{2} \sqrt{4+x}\, \sqrt{1+\left(\frac{1}{2\sqrt{4+x}}\right)^2}\, dx = 2\pi \int_{-3}^{2} \cancel{\sqrt{4+x}} \cdot \sqrt{\frac{4(4+x)+1}{4\cancel{(4+x)}}}\, dx =$$

$$= 2\pi \int_{-3}^{2} \sqrt{\frac{16+4x+1}{4}}\, dx = \cancel{2}\pi \cdot \cancel{\frac{1}{2}} \int_{-3}^{2} \sqrt{4x+17}\, dx \quad \left| \begin{array}{l} 4x+17 = t \\ 4\,dx = dt \\ dx = \frac{1}{4}\,dt \end{array} \quad \begin{array}{c|c|c} x & -3 & 2 \\ \hline t & 5 & 25 \end{array} \right|$$

$$= \frac{1}{4}\pi \int_{5}^{25} \sqrt{t}\, dt = \frac{1}{4}\pi \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right]_5^{25} = \frac{\pi}{6} \left[t\sqrt{t} \right]_5^{25} = \frac{\pi}{6}\left(25 \cdot 5 - 5\sqrt{5}\right) =$$

$$= \underline{\underline{\frac{5}{6}\pi\left(25 - \sqrt{5}\right)}}$$

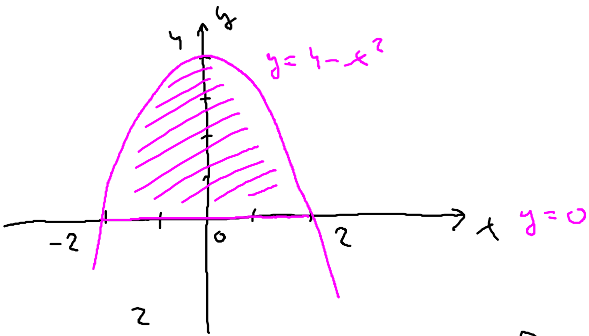
Příklad 5.8 Určete těžiště homogenní oblasti o konstantní plošné hustoitě. Oblast je ohraničená křivkami $y = 4 - x^2$, $y = 0$.

TĚŽIŠTĚ HOMOGENNÍ OBLASTI (O PLOŠNÉ HUSTOTĚ σ kg/m³)

$$T = [x_T, y_T] = \left[\frac{S_y}{m}, \frac{S_x}{m}\right]$$

$$m = \sigma \int_a^b [f(x) - g(x)]\, dx \quad \dots \text{HMOTNOST}$$

$$\left.\begin{aligned} S_x &= \frac{1}{2}\sigma \int_a^b [f^2(x) - g^2(x)]\, dx \\ S_y &= \sigma \int_a^b x\,[f(x) - g(x)]\, dx \end{aligned}\right\} \dots \text{STATICKÝ MOMENT}$$



OBLAST JE SYMETRICKÁ $\Rightarrow x_T = 0$

$$m = \sigma \int_{-2}^{2} (4 - x^2) - 0\, dx = \sigma \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \sigma\left(8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right)\right) =$$

$$= \sigma\left(16 - \frac{16}{3}\right) = \frac{32}{3}\sigma$$

$$S_x = \frac{1}{2}\sigma \int_{-2}^{2} (4 - x^2)^2\, dx = \frac{1}{2}\sigma \int_{-2}^{2} 16 - 8x^2 + x^4\, dx = \frac{1}{2}\sigma\left[16x - \frac{8}{3}x^3 + \frac{x^5}{5}\right]_{-2}^{2} =$$

$$= \frac{1}{2}\sigma\left(32 - \frac{64}{3} + \frac{32}{5}\right) 2 = \dots = \frac{256}{15}\sigma$$

$$y_T = \frac{S_x}{m} = \frac{\frac{\cancel{256}^{8}\cancel{\sigma}}{\cancel{15}_{5}}}{\frac{\cancel{32}}{\cancel{3}}\cancel{\sigma}} = \frac{8}{5} \qquad \Rightarrow \quad \underline{\underline{T\left[0, \frac{8}{5}\right]}}$$

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