Příklad 1.10 $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x$

$$\int \frac{1}{\sqrt{3+2x-x^2}}\,dx = \int \frac{1}{\sqrt{3-(x^2-2x)}}\,dx = \int \frac{1}{\sqrt{3-(x-1)^2+1}}\,dx = \int \frac{1}{4-(x-1)^2}\,dx =$$

$$= \int \frac{dx}{2\sqrt{1-\frac{1}{4}(x-1)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-\left[\frac{1}{2}(x-1)\right]^2}}\,dx \left| \begin{array}{l} t = \frac{1}{2}(x-1) \\ dt = \frac{1}{2}\,dx \\ 2dt = dx \end{array} \right| =$$

$$= \frac{1}{2}\cdot 2\int \frac{dt}{\sqrt{1-t^2}} = \arcsin t = \underline{\underline{\arcsin \frac{x-1}{2} + C}}$$

Příklad 2.4 $\displaystyle\int \frac{x^2-2x-7}{(x^2-2x+1)(x^2+2x+5)}\,\mathrm{d}x$

$$\int \frac{x^2-2x-7}{(x^2-2x+1)(x^2+2x+5)}\,dx \overset{*}{=} \frac{1}{2}\int\frac{dx}{x-1} - \int\frac{dx}{(x-1)^2} + \frac{1}{2}\underbrace{\int\frac{1-x}{x^2+2x+5}\,dx}_{A} = (*)$$

$*$

$$\frac{x^2-2x-7}{(x-1)^2(x^2+2x+5)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2x+5}$$

$$x^2-2x-7 = A(x-1)(x^2+2x+5)+B(x^2+2x+5)+(Cx+D)(x^2-2x+1)$$

$$x^2-2x-7 = Ax^3+Ax^2+3Ax-5A+Bx^2+2Bx+5B+Cx^3-2Cx^2+Cx+Dx^2-2Dx+D$$

$$\begin{aligned}
x^3&:\ 0 = A + C\\
x^2&:\ 1 = A+B-2C+D\\
x^1&:\ -2 = 3A+2B+C-2D\\
x^0&:\ -7 = -5A+5B+D
\end{aligned}
\qquad
\left(\begin{array}{cccc|c}
1&0&1&0&0\\
1&1&-2&1&1\\
3&2&1&-2&-2\\
-5&5&0&1&-7
\end{array}\right)
\sim
\left(\begin{array}{cccc|c}
1&0&1&0&0\\
0&1&-3&1&1\\
0&-1&1&1&1\\
0&5&5&1&-7
\end{array}\right)
\sim$$

$$\left(\begin{array}{cccc|c}
1&0&1&0&0\\
0&1&-3&1&1\\
0&0&1&-1&-1\\
0&0&5&-1&-3
\end{array}\right)
\sim
\left(\begin{array}{cccc|c}
1&0&1&0&0\\
0&1&-3&1&1\\
0&0&1&-1&-1\\
0&0&0&2&1
\end{array}\right)
\qquad
\begin{aligned}
&\Rightarrow A = -C = \tfrac{1}{2}\\
&\Rightarrow B = 1+3C-D = -1\\
&\Rightarrow C = -1+D = -\tfrac{1}{2}\\
&\Rightarrow D = \tfrac{1}{2}
\end{aligned}$$

$$A = \int\frac{1-x}{x^2+2x+5}\,dx = \int\frac{-\frac{1}{2}(2x+2)+2}{x^2+2x+5}\,dx = -\frac{1}{2}\underbrace{\int\frac{2x+2}{x^2+2x+5}\,dx}_{B} + 2\underbrace{\int\frac{dx}{x^2+2x+5}}_{C}$$

$$B = \int\frac{2x+2}{x^2+2x+5}\,dx \left|\begin{array}{l} t = x^2+2x+5\\ dt = (2x+2)\,dx\end{array}\right| = \int\frac{1}{t}\,dt = \ln|t| = \ln(x^2+2x+5)$$

$$C = \int\frac{dx}{x^2+2x+5}\,dx = \int\frac{dx}{(x+1)^2-1+5} = \int\frac{dx}{4+(x+1)^2} = \frac{1}{4}\int\frac{dx}{1+\left[\frac{1}{2}(x+1)\right]^2}\left|\begin{array}{l} t = \frac{1}{2}(x+1)\\ dt = \frac{1}{2}\,dx\end{array}\right|$$

$$= \frac{1}{4}\cdot 2\int\frac{dt}{1+t^2} = \frac{1}{2}\operatorname{arctg} t = \frac{1}{2}\operatorname{arctg}\frac{x+1}{2}$$

$$\underline{\underline{(*) = \frac{1}{2}\ln|x-1| + \frac{1}{x-1} - \frac{1}{4}\ln(x^2+2x+5) + \frac{1}{2}\operatorname{arctg}\frac{x+1}{2} + C}}$$

Příklad 3.5 $\displaystyle\int \frac{3x+2}{\sqrt{x^2+2x+3}}\,\mathrm{d}x$

$$\int \frac{3x+2}{\sqrt{x^2+2x+3}}\,dx = \int \frac{\frac{3}{2}(2x+2)-1}{\sqrt{x^2+2x+3}}\,dx = \frac{3}{2}\underbrace{\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx}_{A} - \underbrace{\int \frac{1}{\sqrt{x^2+2x+3}}\,dx}_{B} \stackrel{(*)}{=}$$

$$A = \int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx \left| \begin{array}{l} t = x^2+2x+3 \\ dt = (2x+2)\,dx \end{array} \right| = \int \frac{dt}{\sqrt{t}} = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3}$$

$$B = \int \frac{1}{\sqrt{x^2+2x+3}}\,dx = \int \frac{dx}{\sqrt{(x+1)^2-1+3}} = \int \frac{dx}{\sqrt{2+(x+1)^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1+\left[\frac{1}{\sqrt{2}}(x+1)\right]^2}} \left| \begin{array}{l} s = \frac{1}{\sqrt{2}}(x+1) \\ ds = \frac{1}{\sqrt{2}}\,dx \\ \sqrt{2}\,ds = dx \end{array} \right| =$$

$$= \frac{1}{\sqrt{2}}\cdot\sqrt{2}\int \frac{ds}{\sqrt{1+s^2}} \left| \begin{array}{ll} \sqrt{1+s^2} = u - s \quad /^2 & \Rightarrow \; u = \sqrt{1+s^2}+s \\ 1+\cancel{s^2} = u^2 - 2us + \cancel{s^2} & u - s = u - \frac{u^2-1}{2u} = \frac{2u^2-u^2+1}{2u} = \frac{u^2+1}{2u} \\ s = \frac{u^2-1}{2u} & \\ ds = \frac{2u\cdot 2u - (u^2-1)2}{4u^2}\,du = \frac{4u^2-2u^2+2}{4u^2}\,du = \frac{u^2+1}{2u^2}\,du & \end{array} \right| =$$

$$= \int \frac{1}{\frac{u^2+1}{2u}}\cdot\frac{u^2+1}{2u^2}\,du = \int \frac{1}{u}\,du = \ln|u| = \ln\left|\sqrt{1+s^2}+s\right| = \ln\left|\sqrt{1+\frac{(x+1)^2}{2}}+\frac{x+1}{\sqrt{2}}\right| =$$

$$= \ln\left|\frac{1}{\sqrt{2}}\sqrt{2+x^2+2x+1}+\frac{x+1}{\sqrt{2}}\right| = \ln\left|\frac{x+1+\sqrt{x^2+2x+3}}{\sqrt{2}}\right|$$

$$(*) = 3\sqrt{x^2+2x+3} - \ln\frac{1}{\sqrt{2}}\left|x+1+\sqrt{x^2+2x+3}\right| + C =$$

$$= 3\sqrt{x^2+2x+3} - \ln\left|x+1+\sqrt{x^2+2x+3}\right| + \underbrace{\ln\frac{1}{\sqrt{2}} + C}_{} =$$

$$= \underline{\underline{3\sqrt{x^2+2x+3} - \ln\left|x+1+\sqrt{x^2+2x+3}\right| + K}}$$