

OPAKOVÁNÍ

A) ROVNICE A NEROVNICE

Příklad 1. Řešte v $\mathbb{R}$:

1) $2x^2 - 3x - 5 = 0$

$$D = (-3)^2 - 4 \cdot 2 \cdot (-5) = 9 + 40 = 49$$

$$x_{1,2} = \frac{-(-3) \pm \sqrt{49}}{2 \cdot 2} = \frac{3 \pm 7}{4} = \begin{cases} x_1 = \frac{3+7}{4} = \frac{10}{4} = \frac{5}{2} \\ x_2 = \frac{3-7}{4} = \frac{-4}{4} = \underline{\underline{-1}} \end{cases}$$

2) Řešení pomocí Viětových vzorců:

$$x^2 + 2x - 15 = 0$$

$$(x-3)(x+5) = 0$$

$$x-3=0 \vee x+5=0$$

$$\underline{\underline{x_1 = 3}}$$

$$\underline{\underline{x_2 = -5}}$$

Řešení pomocí Viětových vzorců: hledáme taková dvě čísla, aby jejich součin byl -15 a jejich součet 2 .

3) $x^2 - |x-1| = 1$

Nulový bod výrazu uvnitř absolutní hodnoty je 1 .

	$(-\infty, 1)$	$(1, -\infty)$
$x-1$	$-$	$+$

• $x \in (-\infty, 1)$

• $x \in (1, -\infty)$

$$x^2 - (-x+1) = 1$$

$$x^2 + x - 1 = 1$$

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2) = 0$$

$$\underline{\underline{x_1 = 1}} \in (-\infty, 1)$$

$$\underline{\underline{x_2 = -2}} \in (-\infty, 1)$$

$$x^2 - (x-1) = 1$$

$$x^2 - x + 1 = 1$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

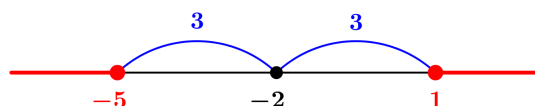
$$x_3 = 0 \notin (1, -\infty)$$

$$x_4 = 1 \notin (1, -\infty)$$

Řešení: $\underline{\underline{x \in \{-2, 1\}}}$

4) $|x+2| \geq 3$

$$|x-(-2)| \geq 3$$



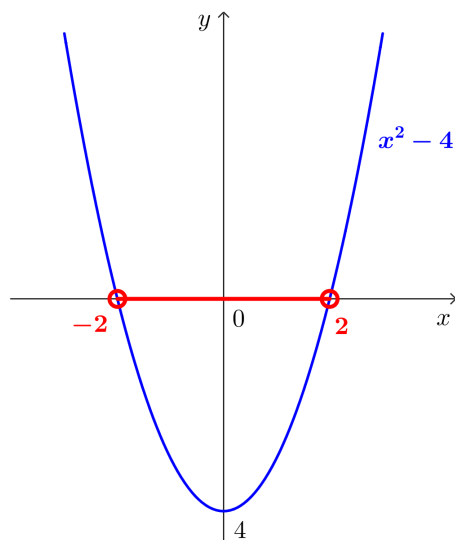
Řešení: $\underline{\underline{x \in (-\infty, -5] \cup [1, \infty)}}$

5) $x^2 < 4$

I. způsob:

$$\begin{array}{lcl} x^2 < 4 & \rightarrow & x^2 = 4 \\ x^2 - 4 < 0 & & x = \pm 2 \end{array}$$

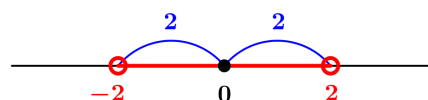
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$$\underline{\underline{x \in (-2, 2)}}$$

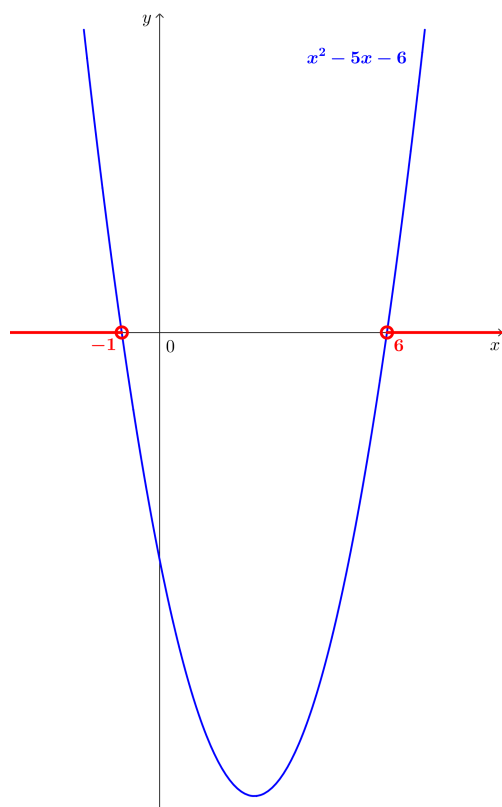
II. způsob:

$$\begin{array}{l} x^2 < 4 \\ |x| < 2 \\ |x - 0| < 2 \end{array}$$



$$6) \quad x^2 - 5x - 6 > 0 \quad \rightarrow$$

↓



$$\underline{\underline{x \in (-\infty, -1) \cup (6, \infty)}}$$

$$\begin{array}{l} x^2 - 5x - 6 = 0 \\ (x + 1)(x - 6) = 0 \\ x_1 = -1 \quad x_2 = 6 \end{array}$$

$$\begin{aligned}
 7) \quad & 4^{x+1} + 8 \cdot 4^x = 12 \\
 & 4 \cdot 4^x + 8 \cdot 4^x = 12 \\
 & 12 \cdot 4^x = 12 \\
 & 4^x = 1 \\
 & 4^x = 4^0 \\
 & \underline{\underline{x = 0}}
 \end{aligned}$$

$$\begin{aligned}
 8) \quad & 25^{-x} + 5^{1-x} = 50 \\
 & (5^{-x})^2 + 5 \cdot 5^{-x} - 50 = 0 \quad \dots \quad \text{substituce: } t = 5^{-x} \\
 & t^2 + 5t - 50 = 0 \\
 & (t - 5)(t + 10) = 0 \\
 & t_1 = 5 \quad \rightarrow \quad 5^{-x} = 5 \\
 & \quad \quad \quad -x = 1 \\
 & \quad \quad \quad \underline{\underline{x = -1}} \\
 & t_2 = -10 \quad \rightarrow \quad \text{nemá řešení, protože } 5^{-x} > 0 \quad \forall x \in \mathbb{R}
 \end{aligned}$$

$$\begin{aligned}
 9) \quad & \log_3(x+1) + \log_3(x+3) = 1 && \text{Podmínky řešitelnosti:} \\
 & \log_3[(x+1)(x+3)] = \log_3 3 && x+1 > 0 \quad \wedge \quad x+3 > 0 \\
 & (x+1)(x+3) = 3 && x > -1 \quad \quad \quad x > -3 \\
 & x^2 + 4x + 3 = 3 \\
 & x^2 + 4x = 0 && x \in (-1, \infty) \\
 & x(x+4) = 0 \\
 & \underline{\underline{x_1 = 0}} \in (-1, \infty) \\
 & x_2 = -4 \notin (-1, \infty)
 \end{aligned}$$

$$\begin{aligned}
 10) \quad & 2^{\ln x} - 4^{\ln x} = 0 \quad \dots \quad \text{Podmínky řešitelnosti: } x \in \mathbb{R}^+ \\
 & 2^{\ln x} - (2^{\ln x})^2 = 0 \quad \dots \quad \text{substituce: } t = 2^{\ln x} \\
 & t - t^2 = 0 \\
 & t(1-t) = 0 \\
 & t_1 = 0 \quad \rightarrow \quad 2^{\ln x} = 0 \quad \dots \quad \text{nemá řešení, protože } 2^y > 0 \quad \forall y \in \mathbb{R} \\
 & t_2 = 1 \quad \rightarrow \quad 2^{\ln x} = 1 \\
 & \quad \quad \quad 2^{\ln x} = 2^0 \\
 & \quad \quad \quad \ln x = 0 \\
 & \quad \quad \quad \underline{\underline{x = 1}} \in \mathbb{R}^+
 \end{aligned}$$

$$\begin{aligned}
 11) \quad & 3 \cotg \frac{x}{2} = -\sqrt{3} \\
 & \cotg \frac{x}{2} = -\frac{\sqrt{3}}{3} \quad \Rightarrow \quad \frac{x}{2} = \frac{2}{3}\pi + k\pi \\
 & \quad \quad \quad \underline{\underline{x = \frac{4}{3}\pi + 2k\pi, k \in \mathbb{Z}}}
 \end{aligned}$$

$$12) \sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{6} = \frac{\pi}{6} + 2k\pi \quad \vee$$

$$\underline{\underline{x_1 = 2k\pi, k \in \mathbb{Z}}}$$

$$x + \frac{\pi}{6} = \frac{5}{6}\pi + 2k\pi$$

$$\underline{\underline{x_2 = \frac{2}{3}\pi + 2k\pi, k \in \mathbb{Z}}}$$

B) ÚPRAVY VÝRAZŮ

Příklad 2. Upravte výrazy na co nejjednodušší tvar:

$$1) \frac{2x}{x+2} - \frac{6x}{6-3x} + \frac{8x}{x^2-4} = \frac{2x}{x+2} - \frac{6x}{3(2-x)} + \frac{8x}{x^2-4} = \frac{2x}{x+2} + \frac{2x}{x-2} + \frac{8x}{x^2-4} =$$

$$= \frac{2x(x-2) + 2x(x+2) + 8x}{x^2-4} = \frac{2x^2 - 4x + 2x^2 + 4x + 8x}{x^2-4} = \frac{4x^2 + 8x}{x^2-4} =$$

$$= \frac{4x(x+2)}{(x+2)(x-2)} = \underline{\underline{\frac{4x}{x-2}}}$$

$$2) \frac{\frac{x}{x-1} - \frac{x+1}{x}}{\frac{x}{x-1} - 1} = \frac{\frac{x^2 - (x+1)(x-1)}{x(x-1)}}{\frac{x - (x-1)}{x-1}} = \frac{\frac{x^2 - (x^2 - 1)}{x(x-1)}}{\frac{x - x + 1}{x-1}} = \frac{x^2 - x^2 + 1}{x} = \underline{\underline{\frac{1}{x}}}$$

$$3) \frac{a^3 - b^3}{a + \frac{b^3}{a+b}} = \frac{(a-b)(a^2 + ab + b^2)}{\frac{a^2 + ab + b^2}{a+b}} = (a-b)(a+b) = \underline{\underline{a^2 - b^2}}$$

$$4) \left(\frac{x-1}{x-2} - \frac{x}{x-1}\right) \cdot \left(x - \frac{x}{x+1}\right) \cdot (x^2 - 1) =$$

$$= \frac{(x-1)^2 - x(x-2)}{(x-1)(x-2)} \cdot \frac{x(x+1) - x}{x+1} \cdot (x+1)(x-1) =$$

$$= \frac{(x^2 - 2x + 1 - x^2 + 2x)(x^2 + x - x)}{x-2} = \underline{\underline{\frac{x^2}{x-2}}}$$

$$5) \frac{\frac{b}{a^2+ab} + \frac{2}{a+b} + \frac{a}{b^2+ab}}{\frac{\frac{a}{b} - \frac{b}{a}}{\frac{a}{b} - \frac{b}{a}}} = \frac{\frac{b}{a(a+b)} + \frac{2}{a+b} + \frac{a}{b(b+a)}}{\frac{\frac{a}{b} - \frac{b}{a}}{\frac{a}{b} - \frac{b}{a}}} = \frac{\frac{b^2 + 2ab + a^2}{ab(a+b)}}{\frac{a^2 - b^2}{ab}} =$$

$$= \frac{\frac{(a+b)^2}{a+b}}{(a+b)(a-b)} = \frac{(a+b)^2}{(a+b)^2(a-b)} = \underline{\underline{\frac{1}{a-b}}}$$

$$\begin{aligned}
 6) \quad & \frac{\frac{1}{x+y} + \frac{x-y}{(x+y)^2}}{1 + \left(\frac{x-y}{x+y}\right)^2} = \frac{\frac{x+y+x-y}{(x+y)^2}}{\frac{(x+y)^2 + (x-y)^2}{(x+y)^2}} = \frac{2x}{x^2 + 2xy + y^2 + x^2 - 2xy + y^2} = \\
 & = \frac{2x}{2x^2 + 2y^2} = \frac{2x}{2(x^2 + y^2)} = \underline{\underline{\frac{x}{x^2 + y^2}}}
 \end{aligned}$$

$$\begin{aligned}
 7) \quad & \frac{\frac{a^2+4}{a} - 2}{\left(\frac{1}{a^2} + \frac{1}{4}\right) \cdot \frac{a^3+8}{a^2+4}} = \frac{\frac{a^2+4-2a}{a}}{\frac{4+a^2}{4a^2} \cdot \frac{(a+2)(a^2-2a+4)}{a^2+4}} = \frac{4a}{\underline{\underline{a+2}}}
 \end{aligned}$$

$$\begin{aligned}
 8) \quad & \frac{\left(ab - \frac{1}{ab}\right)^2}{\left(a + \frac{1}{b}\right)^2 \cdot \left(b - \frac{1}{a}\right)^3} = \frac{\left(\frac{a^2b^2-1}{ab}\right)^2}{\left(\frac{ab+1}{b}\right)^2 \cdot \left(\frac{ab-1}{a}\right)^3} = \frac{\frac{(a^2b^2-1)^2}{a^2b^2}}{\frac{(ab+1)^2}{b^2} \cdot \frac{(ab-1)^3}{a^3}} = \\
 & = \frac{\frac{(ab+1)^2(ab-1)^2}{a^2b^2}}{\frac{(ab+1)^2}{b^2} \cdot \frac{(ab-1)^3}{a^3}} = \underline{\underline{\frac{a}{ab-1}}}
 \end{aligned}$$

$$\begin{aligned}
 9) \quad & \frac{(x^2-y^2)^2}{x^3-y^3} \cdot \frac{x^2+xy+y^2}{x^3+y^3} \cdot \frac{1}{x^2+2xy+y^2} + \frac{\frac{1-x}{x-x^2}}{x-y+\frac{y^2}{x}} = \\
 & = \frac{(x+y)^2(x-y)^2}{(x-y)(x^2+xy+y^2)} \cdot \frac{x^2+xy+y^2}{(x+y)(x^2-xy+y^2)} \cdot \frac{1}{(x+y)^2} + \frac{\frac{1-x}{x(1-x)}}{\frac{x^2-xy+y^2}{x}} = \\
 & = \frac{x-y}{(x+y)(x^2-xy+y^2)} + \frac{1}{x^2-xy+y^2} = \frac{x-y+x+y}{x^3+y^3} = \underline{\underline{\frac{2x}{x^3+y^3}}}
 \end{aligned}$$

$$\begin{aligned}
 10) \quad & \frac{\frac{a^2+b^2}{b} + 2a}{\frac{1}{b} + \frac{1}{a}} + \frac{2b - \frac{a^2+b^2}{a}}{\frac{1}{b} - \frac{1}{a}} = \frac{\frac{a^2+b^2+2ab}{b}}{\frac{a+b}{ab}} + \frac{\frac{2ab-a^2-b^2}{a}}{\frac{a-b}{ab}} = \frac{\frac{(a+b)^2}{b}}{\frac{a+b}{ab}} + \frac{-\frac{(a-b)^2}{a}}{\frac{a-b}{ab}} = \\
 & = a(a+b) - b(a-b) = a^2 + ab - ba + b^2 = \underline{\underline{a^2 + b^2}}
 \end{aligned}$$

Příklad 3. Upravte výrazy na co nejjednodušší tvar:

$$1) \frac{\cos^2 x}{1 + \sin x} = \frac{1 - \sin^2 x}{1 + \sin x} = \frac{(1 + \sin x)(1 - \sin x)}{1 + \sin x} = \underline{\underline{1 - \sin x}}$$

$$2) \frac{\sin 2x}{1 - \cos 2x} = \frac{2 \sin x \cos x}{1 - (\cos^2 x - \sin^2 x)} = \frac{2 \sin x \cos x}{1 - \cos^2 x + \sin^2 x} = \frac{2 \sin x \cos x}{2 \sin^2 x} =$$

$$= \frac{\cos x}{\sin x} = \underline{\underline{\cotg x}}$$

$$3) \frac{1}{1 + \operatorname{tg}^2 x} + \frac{1}{1 + \cotg^2 x} = \frac{1}{1 + \left(\frac{\sin x}{\cos x}\right)^2} + \frac{1}{1 + \left(\frac{\cos x}{\sin x}\right)^2} =$$

$$= \frac{1}{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}} + \frac{1}{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \cos^2 x + \sin^2 x = \underline{\underline{1}}$$

$$4) \frac{\sin x}{1 + \cos x} + \frac{\sin x}{1 - \cos x} = \frac{\sin x(1 - \cos x) + \sin x(1 + \cos x)}{(1 + \cos x)(1 - \cos x)} =$$

$$= \frac{\sin x(1 - \cos x + 1 + \cos x)}{1 - \cos^2 x} = \frac{2 \sin x}{\sin^2 x} = \underline{\underline{\frac{2}{\sin x}}}$$

$$5) \frac{1 + \cos 2x}{1 - \cos 2x} = \frac{1 + \cos^2 x - \sin^2 x}{1 - \cos^2 x + \sin^2 x} = \frac{2 \cos^2 x}{2 \sin^2 x} = \underline{\underline{\cotg^2 x}}$$

$$6) \frac{\operatorname{tg} x}{1 + \operatorname{tg}^2 x} = \frac{\frac{\sin x}{\cos x}}{1 + \frac{\sin^2 x}{\cos^2 x}} = \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}} = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x}} = \sin x \cos x = \underline{\underline{\frac{1}{2} \sin 2x}}$$

$$7) \frac{1 - \cos 2x + \sin 2x}{1 + \cos 2x + \sin 2x} = \frac{1 - \cos^2 x + \sin^2 x + 2 \sin x \cos x}{1 + \cos^2 x - \sin^2 x + 2 \sin x \cos x} = \frac{2 \sin^2 x + 2 \sin x \cos x}{2 \cos^2 x + 2 \sin x \cos x} =$$

$$= \frac{2 \sin x(\sin x + \cos x)}{2 \cos x(\cos x + \sin x)} = \frac{\sin x}{\cos x} = \underline{\underline{\operatorname{tg} x}}$$

$$8) \frac{\sin x - \sin^3 x}{\cos x - \cos^3 x} = \frac{\sin x(1 - \sin^2 x)}{\cos x(1 - \cos^2 x)} = \frac{\sin x \cos^2 x}{\cos x \sin^2 x} = \frac{\cos x}{\sin x} = \underline{\underline{\cotg x}}$$

$$\begin{aligned}
 9) \quad \frac{2 \sin x - \sin 2x}{2 \sin x + \sin 2x} &= \frac{2 \sin x - 2 \sin x \cos x}{2 \sin x + 2 \sin x \cos x} = \frac{2 \sin x(1 - \cos x)}{2 \sin x(1 + \cos x)} = \frac{1 - \cos x}{1 + \cos x} = \frac{\frac{1 - \cos x}{2}}{\frac{1 + \cos x}{2}} = \\
 &= \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \underline{\underline{\operatorname{tg}^2 \frac{x}{2}}}
 \end{aligned}$$

$$\begin{aligned}
 10) \quad \frac{\operatorname{tg} x \cdot \operatorname{tg} 2x}{\operatorname{tg} x - \operatorname{tg} 2x} &= \frac{\frac{\sin x}{\cos x} \cdot \frac{\sin 2x}{\cos 2x}}{\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x}} = \frac{\frac{\sin x}{\cos x} \cdot \frac{2 \sin x \cos x}{\cos^2 x - \sin^2 x}}{\frac{\sin x}{\cos x} - \frac{2 \sin x \cos x}{\cos^2 x - \sin^2 x}} = \\
 &= \frac{\frac{2 \sin^2 x \cos x}{\cos x(\cos^2 x - \sin^2 x)}}{\frac{\sin x(\cos^2 x - \sin^2 x) - 2 \sin x \cos^2 x}{\cos x(\cos^2 x - \sin^2 x)}} = \frac{2 \sin^2 x \cos x}{\sin x \cos^2 x - \sin^3 x - 2 \sin x \cos^2 x} = \\
 &= \frac{2 \sin^2 x \cos x}{-\sin^3 x - \sin x \cos^2 x} = \frac{2 \sin^2 x \cos x}{-\sin x(\sin^2 x + \cos^2 x)} = -2 \sin x \cos x = \underline{\underline{-\sin 2x}}
 \end{aligned}$$