Příklad 5.1. Určete první pardiální derivace funkce $z = f(x, y)$, která je dána implicitně rovnicí

$$\frac{x}{z} = \ln\frac{z}{y}.$$

Řešení.

$$\begin{aligned}
F(x,y,z) &= \frac{x}{z} - \ln\frac{z}{y} \\
F'_x(x,y,z) &= \frac{1}{z} \\
F'_y(x,y,z) &= -\frac{y}{z}\cdot\left(\frac{z}{y}\right)'_y = \frac{1}{y} \\
F'_z(x,y,z) &= -\frac{x}{z^2} - \frac{y}{z}\cdot\frac{1}{y} = -\frac{x}{z^2} - \frac{1}{z} = -\frac{x+z}{z^2}
\end{aligned}$$

$$\begin{aligned}
z'_x &= -\frac{F'_x}{F'_z} = -\frac{\frac{1}{z}}{-\frac{x+z}{z^2}} = \frac{z}{x+z} \\
z'_y &= -\frac{F'_y}{F'_z} = -\frac{\frac{1}{y}}{-\frac{x+z}{z^2}} = \frac{z^2}{y(x+z)}
\end{aligned}$$

Komentář.

$$\begin{aligned}
\left(\frac{x}{z} - \ln\frac{z}{y}\right)'_x &= \left(\frac{x}{z}\right)'_x - \left(\ln\frac{z}{y}\right)'_x = \frac{1}{z}\,(x)'_x - 0 = \frac{1}{z} \\
\left(\frac{x}{z} - \ln\frac{z}{y}\right)'_y &= \left(\frac{x}{z}\right)'_y - \left(\ln\frac{z}{y}\right)'_y \\
&= 0 - \frac{y}{z}\cdot\left(z\,y^{-1}\right)'_y = -y\left(-\frac{1}{y^2}\right) = \frac{1}{y} \\
\left(\frac{x}{z} - \ln\frac{z}{y}\right)'_z &= x(z^{-1})'_z - \frac{y}{z}\cdot\frac{1}{y} = -\frac{x}{z^2} - \frac{1}{z}
\end{aligned}$$